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Temperature Renormalization in the Non Linear Vlasov Problem

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(August 22, 1969)

Ways of improving the representation of the velocity distribution function in the solution of the non-linear Vlasov equation are discussed. Hermite polynomial coefficients for temperature renormalization are derived. Transformation properties of Hermite polynomials necessary to do this are discussed. Analytic expressions for the truncation error of the temperature renormalized functions are compared with the computer results.

Key words: Expansion coefficients; Hermite polynomials; nonlinear; numerical; plasma; truncation error; Vlasov equation.

Introduction

This paper deals with some mathematical aspects of the numerical solution of the non-linear Vlasov equation by means of eigenfunction expansions [1, 2, 3].[1] The model studied is a one-dimensional, electrically neutral plasma undergoing longitudinal oscillations. Periodic boundary conditions are imposed on the electron density. The initial distribution function is given in dimensionless form [8] by the expression,

$$f(x, v, t=0) = (1+\alpha \cos kx)e^{-v^2/2}/\sqrt{2\pi}$$

where α is the non-linearity parameter and $k=2\pi/L$, L being the periodic length. The electric field is given by the Poisson equation and has the form

$$E(x, t) = \int_0^x dx' \int_{-\infty}^{+\infty} f(x', v, t)dv - x.$$

The initial value of the electric field is thus $E(x, t=0) = (\alpha/k) \sin kx$. The ratio (α/k) determines the initial field amplitude and the initial potential energy content of the system.

In applying the eigenfunction expansion method to the numerical solution of nonlinear partial differential equations, it is very important to obtain as much analytic information as possible concerning expansion properties of the solution, the truncation error incurred by neglecting all but a finite number of terms in the expansion and the effect of truncation errors on the validity of the solution. Procedures of questionable validity are those in which the solution is expanded into a set of orthogonal polynomials without some knowledge of convergence properties of the expansion. When there is no proof that the solution exists on the interval over which the equation is being integrated, extreme caution should be exercised in evaluating numerical results. The lack of general methods of dealing with nonlinear partial differential equations makes one's intuition an unreliable guide in judging numerical results.

[1] Figures in brackets indicate literature references at the end of this paper.

Truncation of an expansion with time-dependent coefficients is a particularly difficult problem since the function that was well represented initially by a certain expansion may no longer be adequately represented by the same number of terms at a later time. The usual numerical tests employed in finite difference methods, such as conservation of certain time-independent quantities, may be inapplicable or not very significant. The physically observable quantities like the number density and macroscopic currents are determined by the first few moments of the distribution function of the phase space coordinates, $f(x, v, t)$. In this sense, the first few moments of an eigenfunction expansion are fixed [4] since they must reproduce the observed quantities. For example, conservation of density in phase space and of total energy in the nonlinear Vlasov equation puts conditions on the zeroth, first, and second moments of the Hermite polynomial expansion. Of these, the zeroth is automatically conserved, while the other two do not indicate any trouble until long after the numerical solution has become invalid.

Associated with the question of truncation is the problem of finding a transformation to a different set of eigenfunctions in the course of the calculation which would give a better representation of the function as it evolves with time than the initial set of eigenfunctions. If such a transformation could be automatically found the truncation error would be reduced. This is the problem discussed below.

In section 1 is derived a transformation of the arguments of Hermite polynomials with the aim of reducing the truncation error with a fixed number of terms in the expansion. The transformation is applied to the problem of temperature renormalization in section 2 and the expansion coefficients are explicitly given.

In section 3 error bounds are derived for the truncation and it is shown that for $\frac{\alpha}{k} \leqslant 1$ temperature renormalization does not result in a significant reduction of truncation error.

Comparison of analytic results with those obtained on the computer is given in appendix II.

1. Transformation Properties of Hermite Polynomials

This section deals with certain transformation properties of Hermite polynomials. These properties will be used in evaluating expansion coefficients of the two-stream distribution function and in temperature normalization. A few necessary formulas pertaining to Hermite polynomials [5] will be given in eqs (1.1), (1.2), and (1.3). Hermite polynomials[2] are defined by means of the relation

$$e^{xt-t^2/2} = \sum_{n=0}^{\infty} \frac{He_n(x)t^n}{n!} . \tag{1.1}$$

From eq (1.1) it follows that

$$He_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{2^k k!(n-2k)!} x^{n-2k} \tag{1.2}$$

where $[n/2]$ is the greatest integer.

From the same equation the following relation can be derived

$$x^n = \sum_{k=0}^{[n/2]} \frac{n!}{2^k k!(n-2k)!} He_{n-2k}(x). \tag{1.3}$$

[2] For discussion of differences in notation and generating functions see appendix I.

1.1. Scale Transformation

Given a Hermite polynomial, $He_n\left(\frac{v}{\sqrt{\epsilon}}\right)$, we would like to express it in terms of $\sum_m He_m(v)$. Using eq (1.2) we obtain

$$He_n\left(\frac{v}{\sqrt{\epsilon}}\right)=\sum_{k=0}^{[n/2]}\frac{(-1)^k n!}{2^k k!(n-2k)!}\quad\frac{v^{n-2k}}{\epsilon^{(n-2k)/2}} \tag{1.4}$$

$$=\frac{1}{\epsilon^{n/2}}\sum_{k=0}^{[n/2]}\frac{(-1)^k n!\epsilon^k}{2^k k!(n-2k)!}v^{n-2k}\,.$$

By eq (1.3) we have:

$$He_n\left(\frac{v}{\sqrt{\epsilon}}\right)=\frac{1}{\epsilon^{n/2}}\sum_{k=0}^{[n/2]}\frac{(-1)^k n!\epsilon^k}{2^k k!(n-2k)!}\sum_{l=0}^{[(n-2k)/2]}\frac{(n-2k)!}{2^l l!(n-2k-2l)!}He_{n-2k-2l}(v). \tag{1.5}$$

Let $k+l=m$. We obtain

$$He_n\left(\frac{v}{\sqrt{\epsilon}}\right)=\frac{1}{\epsilon^{n/2}}\sum_{m=0}^{[n/2]}\sum_{k=0}^{m}\frac{(-1)^k\epsilon^k n!}{2^m k!(m-k)!(n-2m)!}He_{n-2m}(v). \tag{1.6}$$

But the summation on k is a binomial expansion of $\frac{1}{m!}(1-\epsilon)^m$.

This leads to the final expression

$$He_n\left(\frac{v}{\sqrt{\epsilon}}\right)=\frac{1}{\epsilon^{n/2}}\sum_{m=0}^{[n/2]}\frac{n!(1-\epsilon)^m}{2^m m!(n-2m)!}He_{n-2m}(v). \tag{1.7}$$

which for $\epsilon=1$ yields the required identity

$$He_n\left(\frac{v}{\sqrt{\epsilon}}\right)=He_n(v),\qquad \epsilon=1.$$

1.2. Change of Origin

Given a Hermite polynomial, $He_n\left(\frac{v-a}{\sqrt{\epsilon}}\right)$, we shall express it in terms of $\sum_m He_m(v)$.

Let

$$v'=\frac{v}{\sqrt{\epsilon}}\quad\text{and}\quad a'=\frac{a}{\sqrt{\epsilon}} \tag{1.8}$$

From eq (1.2) we have

$$He_n(v'-a')=\sum_{k=0}^{[n/2]}\frac{(-1)^k n!}{2^k k!(n-2k)!}(v'-a')^{n-2k} \tag{1.9}$$

$$=\sum_{k=0}^{[n/2]}\sum_{m=0}^{n-2k}\frac{(-1)^k n!}{2^k k!m!(n-2k-m)!}v'^{n-2k-m}(-a')^m.$$

Summation indices in the nested double sum in eq (1.9) can be interchanged to give the following expression:

$$He_n(v'-a')=\sum_{m=0}^{n}\sum_{k=0}^{[(n-m)/2]}\frac{(-1)^k\,(n-m)!}{2^k k!(n-m-2k)!}\,v'^{n-m-2k}\,\frac{n!(-a')^m}{m!(n-m)!}.$$

Using eq (1.2) with n replaced by $n-m$ we obtain

$$He_n(v'-a')=\sum_{m=0}^{n}He_{n-m}(v')\,\frac{n!(-a')^m}{m!(n-m)!}. \tag{1.10}$$

Similarly, we have

$$He_n(v'+a')=\sum_{m=0}^{n}He_{n-m}(v')\,\frac{n!a'^m}{m!(n-m)!}. \tag{1.11}$$

Substituting $\frac{v}{\sqrt{\epsilon}}$ for v' and $\frac{a}{\sqrt{\epsilon}}$ for a' we get with the aid of eq (1.7)

$$He_n\left(\frac{v-a}{\sqrt{\epsilon}}\right)=\sum_{m=0}^{n}\frac{n!(-a)^m\epsilon^{-m/2}}{m!(n-m)!}\,\frac{1}{\epsilon^{(n-m)/2}}\sum_{k=0}^{[(n-m)/2]}\frac{(n-m)!(1-\epsilon)^k}{2^k k!(n-m-2k)!}\,He_{n-m-2k}(v). \tag{1.12}$$

Let $m+2k=p$

$$He_n\left(\frac{v-a}{\sqrt{\epsilon}}\right)=\frac{1}{\epsilon^{n/2}}\sum_{p=0}^{n}\sum_{k=0}^{[p/2]}\frac{n!a^{p-2k}(-1)^{p-2k}(1-\epsilon)^{2k/2}}{(p-2k)!\,2^k k!\,(n-p)!}\,He_{n-p}(v) \tag{1.13}$$

$$=\frac{1}{\epsilon^{n/2}}\sum_{p=0}^{n}\frac{n!(\epsilon-1)^{p/2}}{p!\,(n-p)!}\,He_{n-p}(v)\sum_{k=0}^{[p/2]}\frac{(-1)^k p!\left(\frac{-a}{\sqrt{\epsilon-1}}\right)^{p-2k}}{2^k k!\,(p-2k)!}.$$

The last sum in eq (1.13) is a Hermite polynomial of the argument $\frac{-a}{\sqrt{\epsilon-1}}$ and we obtain:

$$He_n\left(\frac{v-a}{\sqrt{\epsilon}}\right)=\frac{1}{\epsilon^{n/2}}\sum_{p=0}^{n}\frac{n!(\epsilon-1)^{p/2}}{p!\,(n-p)!}\,He_{n-p}(v)He_p\left(\frac{-a}{\sqrt{\epsilon-1}}\right). \tag{1.14}$$

Similarly, we have

$$He_n\left(\frac{v+a}{\sqrt{\epsilon}}\right)=\frac{1}{\epsilon^{n/2}}\sum_{p=0}^{n}\frac{n!(\epsilon-1)^{p/2}}{p!\,(n-p)!}\,He_{n-p}(v)He_p\left(\frac{a}{\sqrt{\epsilon-1}}\right). \tag{1.15}$$

With (1.14) and (1.15) we find, as we must, that in the limit $a=0$ they reduce to (1·7) and for $\epsilon=1$ they give eqs (1.10) and (1.11) respectively. It should be noted that imaginary values of $\sqrt{\epsilon-1}$ for $\epsilon<1$ in the argument of He_p are immaterial to the analysis since the product $(\epsilon-1)^{p/2}He_p\left(\frac{a}{\sqrt{\epsilon-1}}\right)$ is always real. This can be seen by expressing He_p as in eq (1.1).

2. Temperature Renormalization

In the course of time the initial potential energy of a collisionless plasma is converted into kinetic energy by the Landau damping [6] mechanism. The velocity distribution function initially given by [1]

$$f(v,\, t=0)=\frac{1}{\sqrt{2\pi}}He_0{}^{-v^2/2}. \tag{2.1}$$

will change to reflect the increase in kinetic energy. The wings of the function will rise and the rise will be given by the set of Hermite polynomials $\{C_nHe_n(v)\}$. It must be noted that since no collisions are present, the system will not be in thermal equilibrium at $t \neq 0$ and consequently cannot be represented as it was initially, as a product of a Maxwellian and He_o. An exponential which decreases less rapidly with increasing v may however represent the velocity distribution function better than the set $\{C_nHe_n(v)e^{-v^2/2}\}$. We shall let

$$f(v,\, t)=\frac{1}{\sqrt{2\pi}}\sum_n \bar{C}_n(t)He_n(v)e^{-v^2/2\epsilon(t)} \tag{2.2}$$

where the value of $\epsilon(t)$ is determined by the increase in kinetic energy at the time t and $\bar{C}_n$ is the set of coefficients associated with the new envelope function $e^{-v^2/2\epsilon(t)}$. The new representation may be more rapidly convergent and hence require fewer terms for a given truncation error, thus reducing the computer time, or generate a smaller truncation error for the same number of terms.

We shall give below a transformation which will permit the use of the modified exponential of eq (2.2) for the envelope of the velocity distribution function. Let us assume that the velocity distribution function, at a given time t, is given by the expressions

$$\begin{aligned} f(v,\, t) &= \sum_{m=0}^{M} C_m(t)\, He_m(v)e^{-\frac{v^2}{2\epsilon(\tau)}} \\ f(v,\, t) &= \sum_{n=0}^{N} \bar{C}_n(t)He_n(v)e^{-\frac{v^2}{2\epsilon(t)}} \end{aligned} \tag{2.3}$$

where $\epsilon(\tau)$ refers to the value of ϵ calculated at some previous time τ and $\epsilon(t)$ reflects the current increase in kinetic energy. Thus we get

$$\bar{C}_n=\frac{1}{\sqrt{2\pi}\, n!}\sum_{m=0}^{M} C_m(t)\int_{-\infty}^{\infty} He_n(v)He_m(v)e^{-\frac{v^2}{2\epsilon(\tau)}+\frac{v^2}{2\epsilon(t)}-\frac{v^2}{2}}\, dv. \tag{2.4}$$

Let $$\frac{\epsilon(\tau)\epsilon(t)+\epsilon(t)-\epsilon(\tau)}{\epsilon(\tau)\epsilon(t)}=\gamma \text{ and}$$

let

$$v\sqrt{\gamma}=v'.$$

Then

$$\bar{C}_n(t)=\frac{1}{\sqrt{2\pi n!}}\frac{1}{\sqrt{\gamma}}\sum_{m=0}^{M} C_m\int_{-\infty}^{\infty} He_n\left(\frac{v'}{\sqrt{\gamma}}\right)He_m\left(\frac{v'}{\sqrt{\gamma}}\right)e^{-v'^2/2}dv' \tag{2.5}$$

and conversely we have

$$C_m(t)=\frac{1}{\sqrt{2\pi m!}}\frac{1}{\sqrt{\beta}}\sum_{n=0}^{N} \bar{C}_n(t)\int_{-\infty}^{\infty} He_n\left(\frac{v'}{\sqrt{\beta}}\right)He_m\left(\frac{v'}{\sqrt{\beta}}\right)e^{-v'^2/2}dv'. \tag{2.6}$$

In the above equation

$$\beta=\frac{\epsilon(t)\epsilon(\tau)+\epsilon(\tau)-\epsilon(t)}{\epsilon(t)\epsilon(\tau)}$$

and

$$v\sqrt{\beta}=v'.$$

Let us assume that the C_m are known. Using eq (1.7) to express $He_n\left(\frac{v'}{\sqrt{\gamma}}\right)$ in terms of $He_n(v')$ we obtain with the use of orthogonality conditions the following expression for $\bar{C}_n(t)$:

$$\bar{C}_n(t)=\frac{1}{\sqrt{\gamma}}\sum_{m=0}^{M}\sum_{k=0}^{[m/2]}\sum_{l=0}^{[n/2]}C_m(t)\,\frac{1}{\gamma^{(m+n)/2}}\,\frac{m!\,(1-\gamma)^{k+l}}{2^{k+l}k!\,(m-2k)!\,l!\,(n-2l)!}\,\delta_{n-2l,\,m-2k}. \tag{2.7}$$

The orthogonality condition requires that $2k=m-n+2l$ and eq (2.7) reduces to

$$\bar{C}_n(t)=\frac{1}{\sqrt{\gamma}}\sum_{m=0}^{M}C_m(t)\sum_{l=0}^{[n/2]}\frac{m!\,(1-\gamma)^{[(m-n)/2]+2l}}{\gamma^{(m+n)/2}2^{(m-n+4l)/2}((m-n+2l)/2)!\,((n-2l)!)^2 l!} \tag{2.8}$$

with the negative factorial being defined as zero. The double sum in eq (2.8) can be written in a different form in which no negative factorials occur:

$$\bar{C}_n(t)=\frac{1}{\sqrt{\gamma}}\sum_{l=0}^{[n/2]}\sum_{m=n-2l}^{M}C_m(t)\,\frac{m!\,(1-\gamma)^{[(m-n)/2]+2l}}{\gamma^{(m+n)/2}2^{(m-n+4l)/2}((m-n+2l)/2)!\,((n-2l)!)^2 l!} \tag{2.9}$$

It is very difficult to make any comparison between $\bar{C}_n$ and C_m without the knowledge of the actual values of C_m. Although a bound can be given for $\bar{C}_n$ in terms of C_0 under the assumption that the C_m are monotonically decreasing with increasing m, this bound is of no practical value because it is too generous.

We shall now turn to the simpler problem of calculating the coefficients in the expansion

$$e^{-v^2/2\epsilon}=\sum_{n=0}^{\infty}\bar{C}_n He_n(v)\,e^{-v^2/2}. \tag{2.10}$$

In this case we have

$$\epsilon(t)=1;\qquad \epsilon(\tau)=\epsilon;\qquad \gamma=\frac{1}{\epsilon}$$

and

$$C_0=1;\qquad C_m=0,\qquad m\neq 0. \tag{2.11}$$

Thus the result in eq (2.9) applies to this case for the values of γ and C given above. Because the functions in eq (2.10) are symmetric in v, only Hermite polynomials with even indices will be present in the expansion. Substituting the values of γ and C_m into eq (2.9) we obtain:

$$\bar{C}_{2q}=\frac{1}{\sqrt{\gamma}}\,\frac{1}{\gamma^q}\,\frac{(1-\gamma)^q}{2^q q!}. \tag{2.12}$$

3. Error Bounds for the Expansion

We can obtain an error bound on the truncation of the expansion after the first N terms. Using the asymptotic expression for Hermite polynomials [7] (see appendix I, eq (I.6)) we obtain:

$$\bar{C}_{2q}He_{2q}(v) \sim \frac{e^{v^2/4}\Gamma(2q+1)}{2^q\Gamma(q+1)} \frac{\sqrt{\epsilon}}{q!} \left(\frac{\epsilon-1}{2}\right)^q. \tag{3.1}$$

Using Stirling's formula eq (3.1) reduces to

$$\bar{C}_{2q}He_{2q}(v) \sim \sqrt{\frac{\epsilon}{\pi}}\, e^{v^2/4} \frac{(\epsilon-1)^q}{q^{1/2}}. \tag{3.2}$$

The truncation error is given by the following expression:

$$R_{N+1} \cong \sum_{p=0}^{\infty} \sqrt{\frac{\epsilon}{\pi}}\, e^{v^2/4} \frac{(\epsilon-1)^{N+1+p}}{(N+1+p)^{1/2}}. \tag{3.3}$$

We know that

$$\frac{(\epsilon-1)^{N+1+p}}{(N+1+p)^{1/2}} \leqslant \frac{(\epsilon-1)^{N+1+p}}{(N+1)^{1/2}} \tag{3.4}$$

for all values of p in eq (3.3) because $\epsilon > 1$.

$$R_{N+1} < \sqrt{\frac{\epsilon}{\pi}}\, e^{v^2/4} \frac{(\epsilon-1)^{N+1}}{(N+1)^{1/2}(2-\epsilon)} \tag{3.5}$$

where the summation over p in eq (3.3) has been performed using (3.4).

Let us now estimate a representative value of ϵ that may occur in the solution of non-linear Vlasov equation. From the introduction we recall that the initial distribution function is given by the expression

$$f(x, v, t=0) = (1+\alpha \cos kx)e^{-v^2/2}/\sqrt{2\pi} \tag{3.6}$$

where $k=2\pi/L$ and L is the length of the system. The initial kinetic energy content of the system is given by the integral

$$K = \int_0^L dx \int_{-\infty}^{\infty} v^2 f(x, v, t=0)dv = L. \tag{3.7}$$

The initial potential energy content is given by

$$P = \int_0^L E^2(x)dx = \int_0^L \frac{\alpha^2}{k^2} \sin^2 kx\, dx \tag{3.8}$$

In eq (3.8) the electric field was obtained from eq (3.7) with the aid of Poisson's equation in dimensionless form. Integrating eq (3.8) we obtain the initial potential energy content

$$P = \frac{\alpha^2}{2k^2} L \tag{3.9}$$

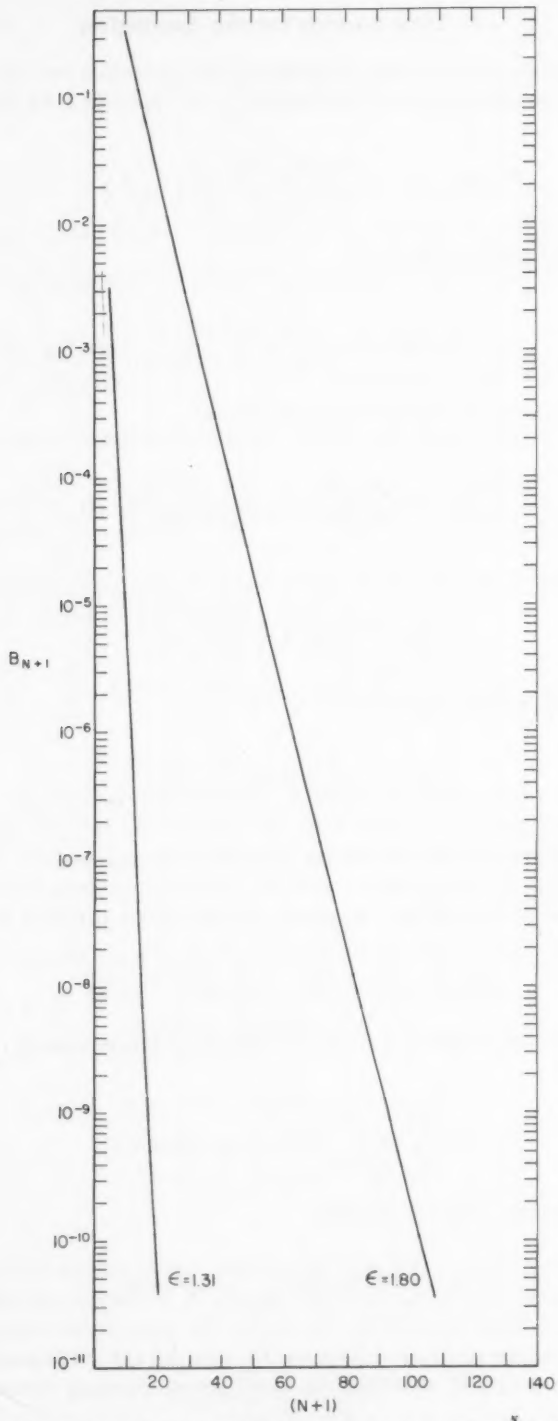

FIGURE 1. *Truncation error bound* B_{N+1} *versus* $(N+1)$ *for the function* $e^{-v^2/2\epsilon} = \sum_{n=0}^{N} C_{2n} He_{2n}(v)$ *for* $v = 2.0$, $\epsilon = 1.31$ *and* $\epsilon = 1.80$.

Choosing $\alpha=0.2$ and $k=0.2$ we obtain

$$P=0.5L \tag{3.10}$$

After the electric field has been completely damped out and the initial potential energy of the field converted to kinetic energy, ϵ can be calculated from the equation

$$1.5=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} v^2 e^{-v^2/2\epsilon}dv \tag{3.11}$$

For the values of α and k given above we obtain the value of ϵ

$$\epsilon=1.31037. \tag{3.12}$$

Figure 1 is a plot of the bound B_{N+1} on the truncation error, R_{N+1}, as a function of the number of terms, N, for $v=2.0$, $\epsilon=1.31$. Figure 2 is a comparison of the number of terms necessary to represent the function given in eq (2.10) with a truncation error $R_{N+1}<10^{-5}T$ obtained analytically and numerically (see appendix II). The value of velocity chosen for the graph is $v=2$. As can be seen by comparing the analytic and the computer curves, the values of ϵ close to 1 are in error because the asymptotic form of Hermite polynomials does not apply. One thing that can be concluded from the graphs, however, is that the truncation error cannot be reduced appreciably by reexpressing the solution of the Vlasov equation by a function of the type given in eq (2.10) for the ratio $\alpha/k\leq 1$. For ratios $\alpha/k>1$, the potential energy content of the system will lead to a larger increase in kinetic energy as the electric field is damped and a transformation of the type discussed would result in a better representation of the solution.

4. Appendix I

There are two different kinds of Hermite polynomials in the literature. One of these is based on the generating function

$$e^{2xt-t^2}=\sum_{n=0}^{\infty}\frac{H_n(x)t^n}{n!} \tag{I.1}$$

and the other, used in this paper, is usually designated in the literature by *He* [6]. It is based on the generating function

$$e^{xt-t^2/2}=\sum_{n=0}^{\infty}\frac{He_n(x)t^n}{n!}. \tag{I.2}$$

The asymptotic expression in the literature [7] is given in the form

$$\Gamma(\tfrac{1}{2}n+1)\exp(-\tfrac{1}{2}x^2)\,H_n(x)=\Gamma(n+1)\,[\cos(N^{1/2}x-\tfrac{1}{2}n\pi)+0(n^{-1/2})] \tag{I.3}$$

where $N=2n+1$.

There is a simple transformation that connects the two polynomials. Let

$$t=\frac{t'}{\sqrt{2}}\quad\text{and}\quad x=\frac{x'}{\sqrt{2}}. \tag{I.4}$$

In eq (I.1) we obtain:

$$e^{x't'-t'^2/2}=\sum_{n=0}^{\infty}\frac{H_n\left(\frac{x'}{\sqrt{2}}\right)t'^n}{2^{n/2}n!}=\sum_{n=0}^{\infty}\frac{He_n(x')t'^n}{n!}. \tag{I.5}$$

The last equality follows from eq. (I.2).

Using the transformed variable of eq (I.5), the asymptotic form for $\mathrm{He}_n(x)$ is

$$He_n(x)=\frac{e^{x^2/4}}{2^{n/2}}\frac{\Gamma(n+1)}{\Gamma(\frac{1}{2}n+1)}\left[\cos\left(N^{1/2}\frac{x}{\sqrt{2}}-\frac{1}{2}n\pi\right)+0\,(n^{-1/2})\right] \tag{I.6}$$

where again $N=2n+1$.

5. Appendix II

A computer program was written in Fortran to calculate the terms of the expansion of $f(v, t)$ of eq (2.10). The $He_{2n}(v)$ and C_{2n} were calculated recursively. Scaled polynomials, He_{2n}, were used to stay within the range of the computer in single precision.

$$\bar{H}e_{2n}(v)=\frac{1}{\sqrt{(2n)!}}He_{2n}(v) \tag{II.1}$$

$$\bar{C}_{2n}=\sqrt{(2n)!}\,C_{2n}.$$

A partial sum, P_n, of the first n terms was formed and more terms were added until the relative error ratio test of eq (II.2) was satisfied for one hundred successive terms.

$$|\mathrm{R.E.}(P_n)|=\frac{|P_{n+1}-P_n|}{|P_n|}<10^{-5}. \tag{II.2}$$

Using eqs (3.2), (3.5), and (II.2) the three term inequality relating R_{N+1} to the truncation error bound 10^{-5} is:

$$R_{N+1}<\frac{\sqrt{\frac{\epsilon}{\pi}}\,e^{x^2/4}\,(\epsilon-1)^{N+1}}{\sqrt{N+1}\,(2-\epsilon)}<10^{-5}T \tag{II.3}$$

$$T=\frac{\sqrt{N}\,(\epsilon-1)}{\sqrt{N+1}\,(2-\epsilon)}fe^{x^2/2}$$

where f is the function of eq (2.10). The parameter values investigated were

$\epsilon=1.10,\ 1.20,\ 1.40,\ 1.60,\ 1.80$
$v=0.00$ to 6.00 in intervals of 0.25.

Figure 2 contains a computer curve for $v=2.0$ and a comparison curve based on eq (II.3).

It should be noted that a uniform bound on the truncation error can be obtained by allowing v to assume its largest value.

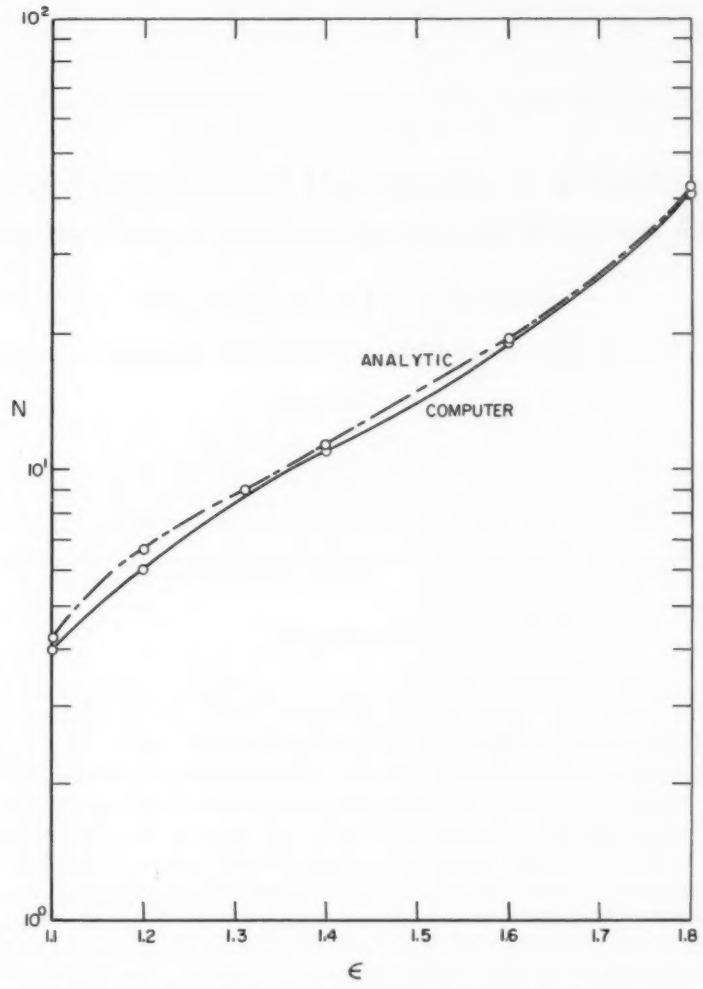

FIGURE 2. N *versus* ϵ *for* $\nu = 2.0$ *for temperature renormalization curve.*

6. References

[1] Sadowski, W. L., On Some Aspects of the Eigenfunction Expansion of the Solution of the Non-Linear Vlasov Equation, Symposium on Computer Simulation of Plasma and Many-Body Problems, NASA, SP–153, 433–440, (1967).

[2] Armstrong, T. P., Numerical Studies of the Non-Linear Vlasov Equation, Ph. D. Thesis, Univ. of Iowa, 66–34 (1966).

[3] Feix, M., Engleman, F., Minardi, E., and Oxenius, J., Non-linear Effects from Vlasov's Equation, Phys. Fluids, 6, 266–275, (1963).

[4] Grad, H., On the Kinetic Theory of Rarefied Gases, Comm. Pure and Appl. Math. **2**, 331–407 (1949).

[5] Rainville, E. D., Special Functions, 187–194 (Macmillan Co., N.Y., 1960).

[6] Dawson, J., On Landau Damping, Phys. Fluids **4**, 869–874 (1961).

[7] Bateman Manuscript Project, Cal. Inst. Technol., Higher Transcendental Functions, A. Erdelyi, editor, Vol. 2, 201 (McGraw-Hill Book Co., N. Y., 1953).

[8] Knorr, G., Numerische Integration der Nichtlinearen Vlasov Gleichung, Z. Naturforsch. **16a**, 1320–1328 (1961).

(Paper 73B4–307)

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Selective Excitation of Harmonics in a Collisionless Plasma by Two Counterstreaming Electron Beams

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(August 22, 1969)

The possibility of selective harmonic excitation in a one-dimensional plasma that obeys the Vlasov equation is discussed. Electron beams, collimated in velocity space, are used for this purpose. Conditions on the velocity distribution function to achieve selective excitation are given. The velocity distribution function is expanded in Gram-Charlier series and the expansion coefficients are given. Bounds on truncation errors of the expansion are derived and compared with computer results.

Key words: Electron trapping; non-linear; numerical; selective harmonic excitation; two-stream; velocity distribution; Vlasov equation.

Introduction

This paper deals with the feasibility of a numerical study of higher harmonic excitation of the non-linear Vlasov plasma by two counterstreaming electron beams. In section 1 it is shown that an electron beam with a high degree of collimation in velocity space can couple to the phase velocity of one of the higher harmonics. The coupling should provide preferential excitation of the harmonic. The collimation is achieved by expressing the velocity distribution function in the form $\exp[-(v\pm a)^2/2\epsilon]$ with ϵ being the collimation parameter that determines the halfwidth of the velocity distribution curve. The distribution function is expanded in Gram-Charlier series and the expansion coefficients are derived in section 2. The necessary transformations of Hermite polynomials are given in the paper "Temperature Renormalization in the Non-Linear Vlasov Problem," immediately preceding this paper. In section 3 are discussed convergence properties of the Gram-Charlier expansion and a criterion is given that allows one to calculate an upper bound on the number of terms necessary to represent the initial distribution function to the desired degree of accuracy.

Comparison of analytic results with those obtained on the computer are discussed in the appendix.

1. Selective Excitation of Harmonics

Numerical work on the solution of the non-linear Vlasov equation by the method of eigenfunction expansion performed by one of the authors [1] [1] brought out some interesting facts associated with the non-linear character of the equation. One of these is that the ratio of plasma frequencies of successive harmonics is not an integer. For example for $k=0.5$

$$\omega_2/\omega_1 = 1.86 \tag{1.1}$$

[1] Figures in brackets indicate the literature references at the end of this paper.

where k is the wave number of the fundamental, $k=2\pi/L$. The phase velocity of a standing wave in a plasma is given by

$$v_{ph}=\frac{\omega_p}{k} \tag{1.2}$$

with ω_p being the plasma frequency. Since the ratio of k's in a periodic system is an integer, we can see from eqs (1.1) and (1.2) that the phase velocities of the fundamental and the second harmonic do not coincide. In the example quoted above the phase velocities of the fundamental and the second harmonic are respectively

$$v_{ph1}=2.71 \qquad \text{and} \qquad v_{ph2}=2.52.$$

The difference in phase velocities is $\Delta v=0.19$. This fact suggests the possibility of a computer study of selective excitation of harmonics by two counterstreaming beams of electrons traveling through the plasma. The electron beams must be sufficiently well collimated in velocity space so as to couple to the desired harmonic. The amount of trapping as a function of the distance from the phase velocity was given by J. Dawson [4]. The functional dependence is rather complicated and will not be given here. It is simpler to use numerical results to examine the possibility of selective excitation of harmonics. Figure 1 shows $f(v)$ that satisfies the non-linear Vlasov equation[2] for $k=0.5$, $\alpha=0.1$ at the dimensionless time $t=32$.

The amplitude of the second harmonic was at all times at least an order of magnitude less than that of the fundamental. The width of the shoulder as can be seen from the insert in figure 1 is about 1.54, with the phase velocities of the fundamental and the second harmonic located on this shoulder. Thus if an electron beam with a velocity distribution with a halfwidth of the order of separation of the two phase velocities were centered on the phase velocity of the second harmonic, the interaction of the second harmonic with the electron beam would excite this harmonic. The effect of the beam on the fundamental would be much less pronounced because of the lower density at the phase velocity of the fundamental.

Since the initial distribution function [1] was chosen to be Maxwellian, a natural choice for the two-stream distribution function is also a Maxwellian plus a combination of Maxwellians centered at a and $-a$.

$$f(v,\, t=0)=\frac{1}{\sqrt{2\pi}(1+2\gamma\sqrt{\epsilon})}\left[e^{-\frac{v^2}{2}}+\gamma e^{-\frac{(v-a)^2}{2\epsilon}}+\gamma e^{-\frac{(v+a)^2}{2\epsilon}}\right] \tag{1.3}$$

where $\dfrac{1}{\sqrt{2\pi}(1+2\gamma\sqrt{\epsilon})}$ is the normalization constant. The parameter ϵ determines the halfwidth of the two-stream distribution function. To obtain a halfwidth of Δv, ϵ must satisfy the relation

$$\frac{(\Delta v)^2}{2\epsilon}=1. \tag{1.4}$$

For $\Delta v=0.19$ this gives the value

$$\epsilon=1.81\times 10^{-2}.$$

[2] Unpublished results.

The individual components of the two-stream distribution functions are expressed in the form

$$e^{-\frac{(v-a)^2}{2\epsilon}} = \sum_n C_n He_n(v) e^{-\frac{v^2}{2}}$$
$$e^{-\frac{(v+a)^2}{2\epsilon}} = \sum_n D_n He_n(v) e^{-\frac{v^2}{2}} \tag{1.5}$$

where He_n are Hermite polynomials. The convergence of the expansion, the value of its coefficients, and the number of terms in the expansion will be treated in detail below.

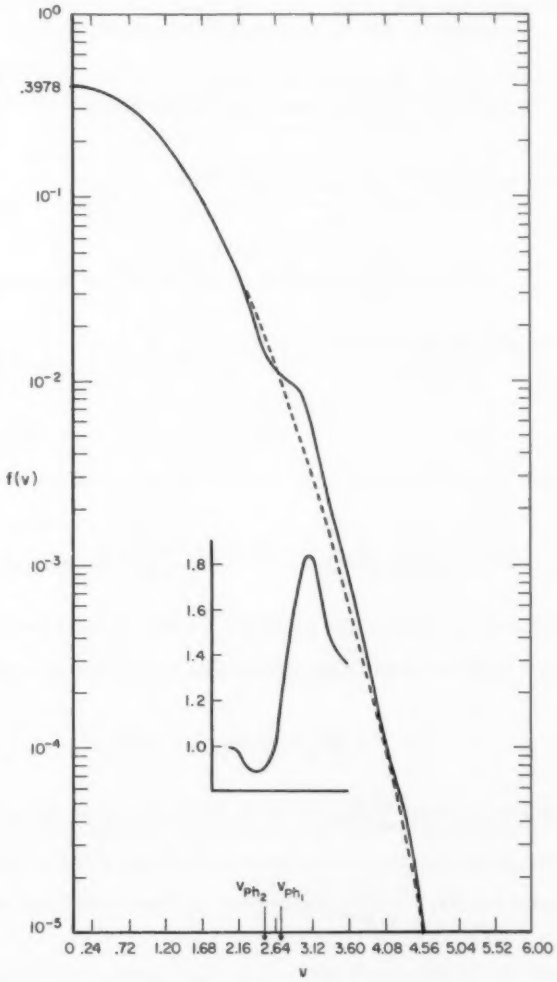

FIGURE 1. f(v) *versus* v *for* k=0.5 *at* t=32.

The dashed line gives $e^{-v^2/2}/\sqrt{2\pi}$ for comparison. The insert represents $f(v)/(e^{-v^2/2}/\sqrt{2\pi})$.

2. Expansion Coefficients for the Two-Stream Problem

We want to express one of the side peaks of the two-stream velocity distribution function of eq (1.5) in the following form:

$$e^{-\frac{(v-a)^2}{2\epsilon}} = g(v)e^{-\frac{v^2}{2}} \tag{2.1}$$

where

$$g(v) = \sum_{n=0}^{\infty} C_n He_n(v). \tag{2.2}$$

Coefficients for the other side peak can be obtained immediately by substituting a for $-a$, as can be easily seen by comparing eqs TR (1.14)[3] and TR (1.15). The expansion coefficients, C_n, are given by the integral:

$$C_n = \frac{1}{\sqrt{2\pi n!}} \int_{-\infty}^{\infty} g(v)e^{-\frac{v^2}{2}} He_n(v)dv. \tag{2.3}$$

Substituting the expression for $g(v)$ from eq (2.1), we obtain

$$C_n = \frac{1}{\sqrt{2\pi n!}} \int_{-\infty}^{\infty} e^{-\frac{(v-a)^2}{2\epsilon}} He_n(v)dv. \tag{2.4}$$

Let us now make the transformation

$$\bar{v} = \frac{v-a}{\sqrt{\epsilon}}. \tag{2.5}$$

This leads to the integral

$$C_n = \frac{1}{\sqrt{2\pi n!}} \frac{1}{\beta^{1/2}} \int_{-\infty}^{\infty} e^{-\bar{v}^2/2} He_n\left(\frac{\bar{v}+\bar{a}}{\sqrt{\beta}}\right) d\bar{v}. \tag{2.6}$$

Here $\beta = \epsilon^{-1}$ was introduced to express the argument of He_n in the form given in eq TR (1.15), and $\bar{a} = \frac{a}{\sqrt{\epsilon}}$. Since $He_0 = 1$ the integrad in eq (2.6) can be considered as a product

$$He_n = He_0 He_n. \tag{2.7}$$

$He_n\left(\frac{\bar{v}+\bar{a}}{\sqrt{\beta}}\right)$ is expressed in terms of $\sum He_n(\bar{v})$ by eq TR (1.15). From the orthogonality conditions we see that the only term in the sum that contributes to the integral is the term with $p = n$.

Substituting this term for $He_n\left(\frac{\bar{v}+\bar{a}}{\sqrt{\beta}}\right)$, integration can be carried out and we obtain:

$$C_n = \frac{1}{n!} \epsilon^{1/2} (1-\epsilon)^{n/2} He_n\left(\frac{a}{\sqrt{1-\epsilon}}\right). \tag{2.8}$$

[3] TR–refers to "Temperature Renormalization in the Non-Linear Vlasov Problem" by the same authors, immediately preceding this paper.

Substituting the values of C_n obtained in eq (2.8) into eq (2.2), we can now express eq (1.3) in terms of its Hermite polynomial expansion.

$$f(v,\, t=0)=\frac{e^{-v^2/2}}{\sqrt{2\pi}\,(1+2\gamma\sqrt{\epsilon})}\left[\,1+\gamma\sum_{n=0}^{\infty}\frac{\epsilon^{1/2}(1-\epsilon)^{n/2}}{n!}He_n\left(\frac{a}{\sqrt{1-\epsilon}}\right)He_n(v)\right.$$
$$\left.+\gamma\sum_{n=0}^{\infty}\frac{\epsilon^{1/2}(1-\epsilon)^{n/2}}{n!}He_n\left(\frac{-a}{\sqrt{1-\epsilon}}\right)He_n(v)\,\right]. \tag{2.9}$$

Since Hermite polynomials with odd indices are odd functions, eq (2.9) reduces to its final form

$$f(v,\, t=0)=\frac{e^{-v^2/2}}{\sqrt{2\pi}\,(1+2\gamma\sqrt{\epsilon})}\left[\,1+2\gamma\sum_{n=0}^{\infty}\frac{\epsilon^{1/2}(1-\epsilon)^{n}}{(2n)!}He_{2n}\left(\frac{a}{\sqrt{1-\epsilon}}\right)He_{2n}(v)\,\right] \tag{2.10}$$

3. Convergence Properties of the Expansion

In numerical work it is desirable to truncate the expansion after the smallest number of terms for a given truncation error. Using the asymptotic form for Hermite polynomials [6] (see TR appendix I for the discussion)

$$He_n(x)=\frac{e^{x^2/4}}{2^{n/2}}\frac{\Gamma(n+1)}{\Gamma(\frac{1}{2}n+1)}\left[\,\cos\left(N^{1/2}\frac{x}{\sqrt{2}}-\frac{1}{2}n\pi\right)+0\,(n^{-1/2})\,\right] \tag{3.1}$$

where $N=2n+1$. For large n, we obtain the asymptotic expression for

$$C_nHe_n(v)\sim\frac{\epsilon^{\frac{1}{2}}}{n!}\frac{(1-\epsilon)^{\frac{n}{2}}}{2^n}\,e^{\frac{v^2}{4}+\frac{a^2}{4(1-\epsilon)}}\,\frac{\Gamma^2(n+1)}{\Gamma^2(\frac{1}{2}n+1)} \tag{3.2}$$

with C_n defined by eq (2.6).

In the above expression the term $0(n^{-1/2})$ has been neglected and $\cos\left(N^{1/2}\,\frac{v}{\sqrt{2}}-\frac{1}{2}\,n\pi\right)$ has been replaced by unity.

Using the well-known relation between the Γ-function and the factorial

$$\Gamma(z+1)=z! \tag{3.3}$$

we obtain the asymptotic expansion for the terms in eq (2.10)

$$C_{2n}He_{2n}(v)\sim e^K\,\frac{\epsilon^{1/2}}{2^{2n}}\,\frac{(2n)!}{(n!)^2}\,(1-\epsilon)^n \tag{3.4}$$

where $K=\frac{v^2}{4}+\frac{a^2}{4(1-\epsilon)}$. Using Stirling's formula for the factorials, we finally get:

$$C_{2n}He_{2n}\sim\frac{\epsilon^{1/2}e^K}{\sqrt{\pi}}\,\frac{(1-\epsilon)^n}{\sqrt{n}}. \tag{3.5}$$

The asymptotic expression for $C_{2n}He_{2n}$ in eq (3.5) tends monotonically to zero with increasing n. This can be shown easily by taking the logarithm of $C_{2n}He_{2n}$

$$\ln C_{2n}He_{2n}\sim\ln B+n\,\ln\,(1-\epsilon)-\tfrac{1}{2}\ln n \tag{3.6}$$

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where

$$B=\frac{\epsilon^{1/2}e^{K}}{\sqrt{\pi}}\,.$$

Since $0<\epsilon\leq 1$, $\ln(1-\epsilon)$ is a negative number and the logarithmic terms in eq (3.6) go to $-\infty$.

Thus

$$\lim_{n\to\infty} C_{2n}He_{2n}=0. \tag{3.7}$$

Let us now find the truncation error introduced into the function by neglecting all the terms in eq (2.10) past the term He_{2N}. This error can be represented in the form:

$$R_{N+1}<\theta\,\frac{\epsilon^{1/2}(1-\epsilon)^{N+1}}{(2N+2)!}\,He_{2N+2}\left(\frac{a}{\sqrt{1-\epsilon}}\right)He_{2N+2}(v)\sum_{k=0}^{\infty}(1-\epsilon)^{k} \tag{3.8}$$

where

$$\theta=2/\sqrt{2\pi}\,(1+2\gamma\sqrt{\epsilon})\,.$$

Expressing all He_{2N+2+p}, $p=0$ to ∞ in their asymptotic form we see that since $1-\epsilon<1$, the geometric progression in eq (3.8) can be summed and we finally obtain a bound on the truncation error

$$R_{N+1}<\theta\,\frac{e^{\frac{v^2}{4}+\frac{a^2}{4(1-\epsilon)}}(1-\epsilon)^{N+1}}{\sqrt{\pi}\,\epsilon^{1/2}(N+1)^{1/2}}\,. \tag{3.9}$$

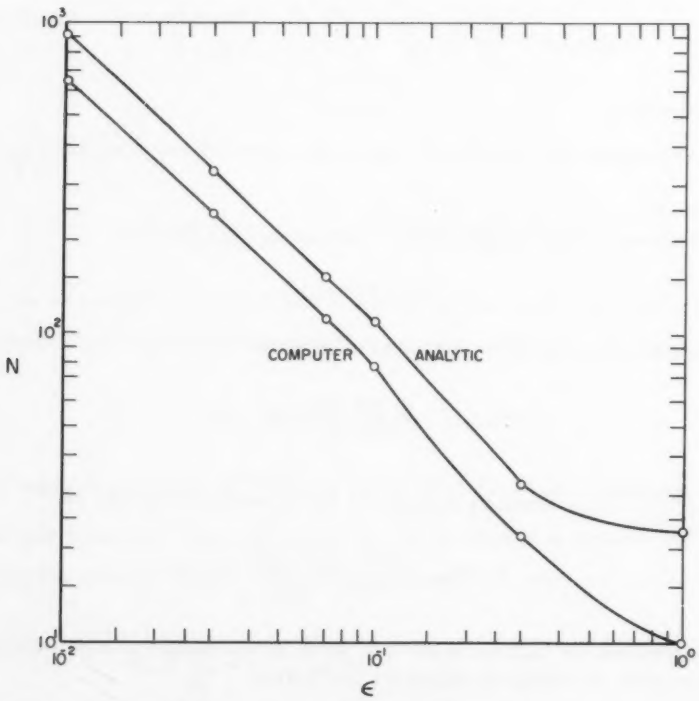

FIGURE 2. N *versus* ϵ *for two-stream function.*

Computer results and analytic results for $v=3.0$, $a=2.0$

The inequality in eq (3.9) makes it possible to calculate N for a given truncation error. This is useful in representing the initial two-stream distribution function in the Vlasov equation. Figure 2 gives the number of terms in the expansion of the expression given in eq (2.10) as a function of ϵ for $a=2.0$ and $v=3.0$ for a truncation error of $R_n < 10^{-5}S$ (see appendix). A graph of the number of terms obtained on the computer is included for comparison. The values of N for $\epsilon=0.9$ and 0.8 do not lie on the curve, which is not surprising since the asymptotic expression for He_n is not very accurate for $N \sim 10$. As N increases the points form a curve which bounds from above the computed curve. This again is not surprising, since the approximations made in eqs (3.2) and (3.8) made the truncation error somewhat bigger than it actually is.

4. Appendix

A computer program was written in Fortran to calculate the terms of the expansion of $f(v, t=0)$ of eq (2.10). The $He_{2n}(v)$ and C_{2n} were calculated recursively. Scaled polynomials, $\bar{H}e_{2n}$, were used to stay within the range of the computer in single precision.

$$\bar{H}e_{2n}(v)=\frac{1}{\sqrt{(2n)!}}He_{2n}(v) \tag{A.1}$$

$$\bar{C}_{2n}=\sqrt{(2n)!}\,C_{2n}.$$

A partial sum, P_n, of the first n terms was formed and more terms were added until the relative error ratio test of eq (A.2) was satisfied for one hundred successive terms.

$$|\text{R.E.}(P_n)|=\frac{|P_{n+1}-P_n|}{|P_n|}<10^{-5}. \tag{A.2}$$

Using eqs (2.10), (3.5), (3.9), and (A.2), the relation of this ratio test to the truncation error is given by eq. (A.3).

$$R_{N+1}<\frac{\theta\, e^{K}(1-\epsilon)^{N+1}}{\epsilon^{1/2}\sqrt{\pi}\,(N+1)^{1/2}}<10^{-5}S \tag{A.3}$$

where

$$S=\frac{\theta}{\epsilon}\frac{fe}{2}^{v^2/2}$$

and f is the un-normalized function calculated on the computer. Figure 2 contains the computer results for $v=3.0$, $a=2.0$ and a comparison analytic curve based on eq (A.3). Other values of the parameters investigated on the computer were

$\epsilon=1.00, 0.30, 0.10, 0.07, 0.03, 0.01$
$a=1.00, 2.00, 3.00, 4.00, 5.00$
$v=0.00$ to 6.00 in intervals of 0.25
$\gamma=0.1$.

Figure 3 contains a set of results for $v=3.0$, $a=1.0$, 3.0, 5.0.

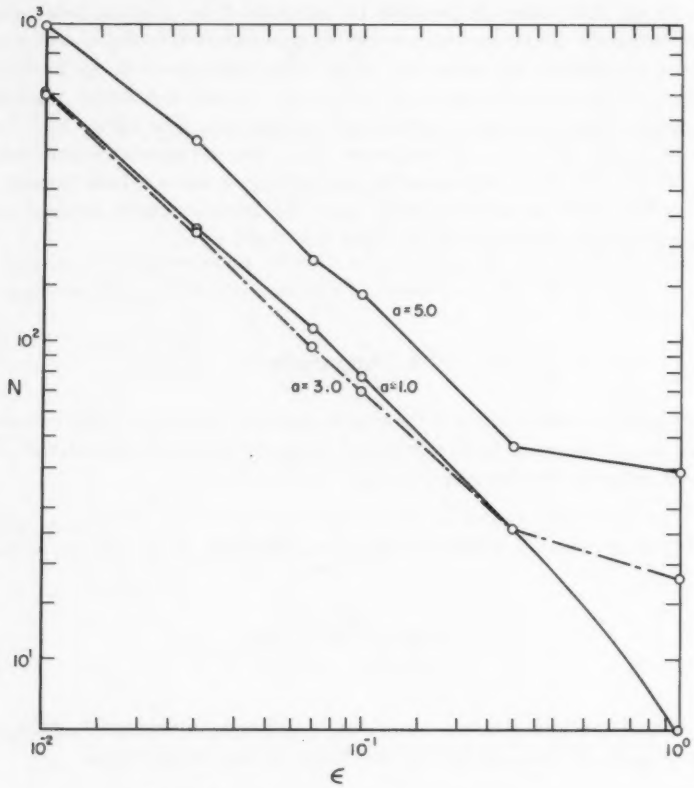

FIGURE 3. N *versus* ϵ *for two-stream function.*

Computer results for $v = 3.0$

5. References

[1] Sadowski, W. L., On Some Aspects of the Eigenfunction Expansion of the Solution of the Non-Linear Vlasov Equation, Symposium on Computer Simulation of Plasma and Many-Body Problems, NASA, SP–153, 433–440, (1967).

[2] Armstrong, T. P., Numerical Studies of the Non-Linear Vlasov Equation, Ph.D. Thesis, Univ. of Iowa, No. 66–34, (1966).

[3] Feix, M., Engleman, F., Minardi, E., and Oxenius, J., Nonlinear effects from Vlasov's equations, Phys. Fluids **6**, 266–275, (1963).

[4] Dawson, J., On Landau damping, Phys. Fluids **4**, 869–874, (1961).

[5] Knorr, G., Numerische Integration der nichtlinearen Vlasov Gleichung, Z. Naturforsch **16a**, 1320–1328, (1961).

[6] Bateman Manuscript Project, Cal. Inst. Technol., Higher Transcendental Functions, A. Erdelyi, editor, Vol. **2**, 201 (McGraw-Hill Book Co., N.Y., 1953).

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Simultaneous Contractification

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(August 8, 1969)

Consider a finite family of continuous self-mappings of a topological space X, with a common fixed point. Suppose that for each member of the family, X has a metric for which that member is a contraction. It is shown that if the family is commutative, then X has a metric under which all members are (simultaneously) contractions. Additional hypotheses are given which ensure the same conclusion in the noncommutative case.

Key words: Contractions; functional analysis: metric spaces; topology.

1. Introduction

This paper deals with continuous self-mappings of a metrizable topological space X. Such a map f is a (ρ, λ)-*contraction* if $\lambda \epsilon (0, 1)$, ρ is some metric on X, and

$$\rho(fx, fy) \leq \lambda \rho(x, y) \qquad (\text{all } x, y \epsilon X). \tag{1.1}$$

We term f *contractifiable* if an appropriate metrization of X makes it a contraction, i.e., if there exists a pair (ρ, λ) such that f is a (ρ, λ)-contraction.

Banach's Contraction Theorem asserts that if f is a (ρ, λ)-contraction for some *complete* metric [2] ρ and some λ, then there is a point $\xi \epsilon X$ and an open neighborhood U of ξ such that [3]

$$f(\xi) = \xi, \tag{1.2}$$

$$f^n(x) \to \xi \qquad (\text{all } x \epsilon X), \tag{1.3}$$

$$f^n(U) \to \{\xi\}. \tag{1.4}$$

The explicit meaning of (1.4) is that for each neighborhood V of ξ, there is an $n(V) > 0$ such that $f^n(U) \subset V$ for all $n \geq n(V)$.

In a previous paper,[4] to be referred to as CONVERSE, the second author proved a converse result: If f satisfies (1.2)–(1.4) for some $\xi \epsilon X$ and some open neighborhood U of ξ, then f is contractifiable. [Moreover, the "contraction constant" λ can be specified to be any assigned member of (0, 1), and the "contractifying" metric ρ can be chosen complete if X admits a complete metric.] Interest in such converses stems from situations in which one would like to apply the Contraction Theorem to the study of some iterative numerical process, but encounters difficulty because for all $\lambda \epsilon (0, 1)$, the associated mapping fails to satisfy (1.1) for the metric initially considered.

Our purpose in the present paper is to extend this converse result to the "simultaneous contractification" of a *family* $\mathfrak{F}$ of maps. We will call such a family *simultaneously contractifiable*

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[2] We call a metric ρ *complete* if (X, ρ) is a complete metric space.

[3] Actually (1.4) is not usually cited as a conclusion of the Contraction Theorem, but is an easy consequence of (1.1) and (1.2).

[4] P. R. Meyers, A Converse to Banach's Contraction Theorem, J. Res. Nat. Bur. Stand. (U.S.), **71B**, (2&3) 73–76 (1967).

if there is a single metric that "works" for all of them, i.e., if X admits a metric ρ under which *all* $f\epsilon\mathfrak{F}$ are contractions. If the associated λ_f can be chosen independent of f, we term $\mathfrak{F}$ *uniformly contractifiable.*

It will be assumed throughout that the members of $\mathfrak{F}$ have a *common* fixed point ξ. This is automatically the case, for example, in the frequently-encountered situations in which the members of $\mathfrak{F}$ *commute.* For a proof, recall that a contractifiable map f with at least one fixed point has a *unique* fixed point ξ_f. For commuting maps f and g, with respective unique fixed points ξ_f and ξ_g, the relation

$$fg(\xi_f) = gf(\xi_f) = g(\xi_f)$$

identifies $g(\xi_f)$ as a fixed point of f, so that (by uniqueness) $g(\xi_f) = \xi_f$. Thus ξ_f is a fixed point of g, yielding the conclusion $(\xi_f = \xi_g)$ of a common fixed point.

For the simplest case, in which $\mathfrak{F}$ is finite and commuting, we can draw the strong conclusion that *individual* contractifiability of the members of $\mathfrak{F}$ implies simultaneous contractifiability . . . in fact, *uniform* contractifiability . . . of the family. The following result will be proven in section 2:

THEOREM 1: *Let $\mathfrak{F}$ be a finite commuting family of continuous self-mappings of* X *which individually satisfy* (1.2)–(1.4). *Then $\mathfrak{F}$ is uniformly contractifiable.*

The following example will show *both* that the finiteness hypothesis cannot be omitted in Theorem 1, *and* that simultaneous contractifiability plus commutativity does not imply uniform contractifiability. Let X be the real line with the usual topology, and take $\mathfrak{F}$ to consist of the infinite family of maps

$$f_i(x) = (1-1/i)x \qquad (i = 1, 2, \ldots).$$

The members of this family commute, have $\xi = 0$ as fixed point, and are all contractions under the standard metric on X. It is easily verified, however, that for *no* open neighborhood U of ξ does

$$f^n(U) \to \{\xi\} \qquad \text{uniformly in } f\epsilon\mathfrak{F}$$

hold. Thus $\mathfrak{F}$ cannot be uniformly contractifiable, else the open unit ball around ξ in a suitable metric would have the property just displayed.

Preparing to drop the commutativity hypothesis in Theorem 1, we define $\mathfrak{F}^0$ to consist of the identity map of X, while $\mathfrak{F}^n$ $(n > 0)$ consists of all n-fold compositions of maps in $\mathfrak{F}$. If $\mathfrak{F}$, with common fixed point ξ, is to be uniformly contractifiable, then the following generalizations of (1.2)–(1.4) must hold:

$$\mathfrak{F}(\xi) = \cup\{f(\xi) : f\epsilon\mathfrak{F}\} = \{\xi\}, \tag{1.2'}$$

$$\mathfrak{F}^n(x) = \cup\{g(x) : g\epsilon\mathfrak{F}^n\} \to \{\xi\} \tag{1.3'}$$

for all $x\epsilon X$, and

$$\mathfrak{F}^n(U) = \cup\{g(U) : g\epsilon\mathfrak{F}^n\} \to \{\xi\} \tag{1.4'}$$

for some open neighborhood U of ξ. It will be shown in section 2 that these necessary conditions, for the uniform contractifiability of $\mathfrak{F}$, are also sufficient when $\mathfrak{F}$ is finite. In other words, the commutativity hypothesis in Theorem 1 can be dropped *if* the hypotheses (1.2)–(1.4), expressing the requirement that individual members of $\mathfrak{F}$ be contractifiable, are replaced by their "uniform in $\mathfrak{F}$" versions (1.2')–(1.4'). Thus we have:

THEOREM 2: *Let $\mathfrak{F}$ be a finite family of continuous self-mappings of* X, *which satisfies* (1.2′)–(1.4′). *Then $\mathfrak{F}$ is uniformly contractifiable.*

(Since (1.2′)–(1.4′) are obtained by systematically replacing f with $\mathfrak{F}$ in (1.2)–(1.4), one might expect that the same replacement would convert the proof of the main theorem in CONVERSE into a proof of Theorem 2. As will be seen, this is actually the case.)

The unresolved problem is that of finding appropriate additional hypotheses to ensure uniform contractifiability (or merely simultaneous contractifiability), with or without commutativity, when $\mathfrak{F}$ is *infinite*.[5]

2. Proofs of Theorems

We first show that Theorem 2 implies Theorem 1. Let $\mathfrak{F}=\{f_1, \ldots, f_m\}$ satisfy the hypotheses of Theorem 1. We shall prove that it also satisfies the hypotheses (1.3′) and (1.4′) of Theorem 2; satisfaction of (1.2′) is immediate since commutativity implies commonness of the fixed point.

As preparation, it will be shown that there is an open neighborhood U of the fixed point ξ such that

$$\mathfrak{F}(U) \subset U \text{ and } \quad \mathfrak{F}^n(U) \to \{\xi\}. \tag{2.1}$$

For this purpose, observe that for $1 \leq i \leq m$ there is an open neighborhood U_i of ξ such that $f_i^n(U_i) \to \{\xi\}$. Let $I=\cap_i^m U_i$; then there are integers $k(i) \geq 1$ such that $f_i^n(U_i) \subset I$ for all $n \geq k(i)$. Now set

$$U=\cap_{j(1)\ldots j(m)} f_1^{-j(1)} \ldots f_m^{-j(m)}(I),$$

where $j(1) \ldots j(m)$ ranges over all integer sequences with $0 \leq j(i) < k(i)$ for $1 \leq i \leq m$. Since $U \subset I$ and $\mathfrak{F}^n(I) \to \{\xi\}$, the second part of (2.1) holds. To prove that the first part also holds, e.g., that $f_1(U) \subset U$, note that

$$f_1(U) \subset f_1^{-j(1)} \ldots f_m^{-j(m)}(I)$$

follows when $j(1) < k(1)-1$ from the consequence

$$f_1(U) \subset f_1^{-[j(1)+1]} \ldots f_m^{-j(m)}(I)$$

of the definition of U, and follows when $j(1)=k(1)-1$ from the definition of $k(1)$ as justifying

$$U \subset f_2^{-j(2)} \ldots f_m^{-j(m)}(I) \subset f_2^{-j(2)} \ldots f_m^{-j(m)}(U_1)$$
$$\subset f_2^{-j(2)} \ldots f_m^{-j(m)} f_1^{-k(1)}(I) = f_1^{-k(1)} f_2^{-j(2)} \ldots f_m^{-j(m)}(I).$$

To demonstrate that $\mathfrak{F}=(f_1, \ldots f_m)$ satisfies (1.3′) and (1.4′), let V be any open neighborhood of ξ. By (2.1), there is an integer K such that $\mathfrak{F}^k(U) \subset V$ for all $k > K$. Let $N=mK+1$. For $n \geq N$, an arbitrary member g of $\mathfrak{F}^n$ can (by commutativity) be written in the form

$$g=f_1^{j(1)} \ldots f_m^{j(m)}, \tag{2.2}$$

where $\Sigma_i j(i)=n \geq N$ and thus $\max_i j(i) > K$. Since $\mathfrak{F}$ is commutative, there is no loss of generality in assuming that the maximum occurs for $i=1$. Then, since $f_i(U) \subset U$ for $i > 1$, we have

$$g(U) \subset f^{j(1)}(U) \subset V.$$

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[5] Some partial results for the commutative case are given in the second author's manuscript, Contractive Semigroups and Uniform Asymptotic Stability, presented at the 6/67 National SIAM Meeting.

This shows that (1.4′) holds. To verify (1.3′), note that for any $x\epsilon X$ there is a $k(x)$ such that $\mathfrak{F}^{k(x)} \subset U$. Let $N' = m[K+k(x)]+1$. Then for $n \geq N'$, an arbitrary $g\epsilon\mathfrak{F}^n$ can be written as (2.2) with $\max_i j(i) > K+k(x)$. Assuming the maximum occurs for $i=1$, we have $j(1)-k(x) > K$ and thus

$$g(x) = f_1^{j(1)-k(x)} f_2^{j(2)} \cdot \cdot \cdot f_m^{j(m)} f_1^{k(x)}(x)$$
$$\epsilon f_1^{j(1)-k(x)} f_2^{j(2)} \cdot \cdot \cdot f_m^{j(m)}(U) \subset f_1^{j(1)-k(x)}(U) \subset V.$$

This shows that (1.3′) holds, completing the proof that Theorem 2 implies Theorem 1.

We turn now to the proof of Theorem 2. Let $\mathfrak{F}$ satisfy the hypotheses of this theorem. As a preliminary, it will be shown that the U in (1.4′) can be assumed to obey

$$\mathfrak{F}(U) \subset U. \tag{2.3}$$

For this purpose, begin with *any* U as in (1.4′). There is an N such that $\mathfrak{F}^n(U) \subset U$ for all $n \geq N$. Let

$$W = \cap_{t=0}^{N-1} \cap \{g^{-1}(U) : g\epsilon\mathfrak{F}^t\}.$$

Then W is an open neighborhood of ξ, and $W \subset U$ so that W satisfies (1.4′). Moreover $\mathfrak{F}(W) \subset W$, as can be seen by considering any $f\epsilon\mathfrak{F}$, any $t\epsilon[0, N-1]$, and any $g\epsilon\mathfrak{F}^t$: that $f(W) \subset g^{-1}(U)$, i.e., $gf(W) \subset U$, follows for $t < N-1$ because of $gf\epsilon\mathfrak{F}^{t+1}$, and for $t = N-1$ because $gf(W) \subset \mathfrak{F}^N(U) \subset U$. Replacing U by W, we can and will assume that the U of (1.4′) satisfies $\mathfrak{F}(U) \subset U$.

Choose any $\lambda\epsilon$ (0, 1) and let ρ_0 be any metric on X, complete if X admits a complete metric. We will construct a metric ρ^* on X, complete if ρ_0 is, such that each $f\epsilon\mathfrak{F}$ is a (ρ^*, λ)-contraction.[6] The construction, which follows closely that in CONVERSE, has three stages. First comes the construction of a metric ρ_M, complete if ρ_0 is, with respect to which each $f\epsilon\mathfrak{F}$ is *nonexpanding*, in the sense that

$$\rho_M(f(x), f(y)) \leq \rho_M(x, y) \qquad (\text{all } x, y\epsilon X).$$

The second stage yields a function d on $X \times X$ which has all the properties desired of ρ^* except perhaps for satisfying the triangle inequality. This is corrected in the third stage, in which $\rho^*(x, y)$ is introduced as what might be called the "d-geodesic distance" between x and y.

The first step is carried out by setting

$$\rho_M(x, y) = \max \{\rho_0(g(x), g(y)) : g\epsilon \cup_0^\infty \mathfrak{F}^n\}. \tag{2.4}$$

That ρ_M is a well-defined metric on X follows as in CONVERSE, and the nonexpansiveness assertion is obvious. Since $\rho_0 \leq \rho_M$, any ρ_M-convergent sequence is also ρ_0-convergent to the same limit, and a ρ_M-Cauchy sequence is also ρ_0-Cauchy. Thus the topological equivalence of the two metrics, as well as the completeness of ρ_M if ρ_0 is complete, will follow once it is shown that for each $x\epsilon X$ and each $\delta > 0$, there is an $\eta < 0$ such that

$$\rho_0(x, y) < \eta \qquad \text{implies} \qquad \rho_M(x, y) < \delta. \tag{2.5}$$

To prove this, observe that (1.3′) assures the finiteness of

$$\nu(x) = \min \{n \geq 0 : \mathfrak{F}^n(x) \subset U\}. \tag{2.6}$$

Since $\mathfrak{F}^{\nu(x)}(x) \subset U$, continuity and the finiteness of $\mathfrak{F}$ imply that for all small enough $\eta > 0$,

$$\rho_0(x, y) < \eta \qquad \text{implies} \qquad \mathfrak{F}^{\nu(x)}(y) \subset U. \tag{2.7}$$

[6] The arbitrariness of λ, and the assertion about completeness, make the results somewhat stronger than was stated in section 1.

Moreover, (1.4′) assures the existence of an N such that

$$\rho_0\text{-diam}\,[\mathfrak{F}^m(U)] < \delta \qquad \text{for} \qquad m > N, \tag{2.8}$$

and continuity and the finiteness of $\mathfrak{F}$ imply that for small enough η,

$$\rho_0(x, y) < \eta \qquad \text{implies} \qquad \rho_0(g(x), g(y)) < \delta \qquad \text{for} \qquad g \in \cup_{n=0}^{N+\nu(x)} \mathfrak{F}^n. \tag{2.9}$$

Suppose η is chosen so small that (2.7) and (2.9) both apply. Consider any n and any $g \in \mathfrak{F}^n$. It will be shown that

$$\rho_0(g(x), g(y)) < \delta, \tag{2.10}$$

thus establishing (2.5). If $n \leq N + \nu(x)$, then (2.10) follows from (2.9). If $n > N + \nu(x)$, write $g = g_2 g_1$ where $g_1 \in \mathfrak{F}^{\nu(x)}$ and $g_2 \in \mathfrak{F}^m$ with $m > N$. By (2.6) and (2.7), $g_1(x)$ and $g_1(y)$ both lie in U, so that $g(x)$ and $g(y)$ both lie in $g_2(U) \subset \mathfrak{F}^m(U)$; now (2.10) follows from (2.8).

To begin the second stage of the construction, let K_n be the closure of $\mathfrak{F}^n(U)$ for $n \geq 0$, and let

$$K_{(-n)} = \mathfrak{F}^{-n}(K_0) = \cap\,\{g^{-1}(K_0) : g \in \mathfrak{F}^n\}.$$

Then $\{K_n\} \to \{\xi\}$, and since $\mathfrak{F}(U) \subset U$, the sequence $\{K_n\}$ is nonascending. Let $n(\xi) = \infty$, and for $x \neq \xi$ set

$$n(x) = \max\,\{n : x \in K_n\} < \infty;$$

then $n(x) \geq 0$ for $x \in K_0$, while for $x \in X - K_0$,

$$n(x) = -\min\,\{k : \mathfrak{F}^k(x) \subset K_0\} < 0.$$

It is easily checked that

$$n(f(x)) \geq n(x) + 1 \qquad \text{for } f \in \mathfrak{F},$$

so that the definitions

$$c(x, y) = \min\,\{n(x),\ n(y)\};\ d(x, y) = \lambda^{c(x, y)} \rho_M(x, y)$$

imply that d has the desired property

$$d(f(x), f(y)) \leq \lambda d(x, y) \qquad (x, y \in X;\ f \in \mathfrak{F}).$$

For the third stage, denote by Σ_{xy} the set of finite chains $\sigma_{xy} = [x = x_0, \ldots, x_t = y]$ between x and y, with associated lengths

$$L(\sigma_{xy}) = \sum_1^t d(x_{i-1}, x_i),$$

and put

$$\rho^*(x, y) = \inf\,\{L(\sigma_{xy}) : \sigma_{xy} \in \Sigma_{xy}\}.$$

That ρ^* has all the desired properties follows exactly as in CONVERSE, with sets $B_\nu = X - f^{-\nu}(U)$ of CONVERSE replaced by

$$B_\nu = X - \mathfrak{F}^{-\nu}(U) = X \cap \{g^{-1}(U) : g \in \mathfrak{F}^\nu\}.$$

(Paper 73B4–309)

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A Theorem on Convex Hulls

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Let p be a point in the interior K° of the convex hull $K=K(S)$ of a bounded point-set S in a real Hilbert space. A quantity $R(p)$ is determined such that every closed ball of radius $>R(p)$, if it contains p, must also meet S.

Key words: Convex geometry; convex sets.

1. Introduction

Let $K=K(S)$ be the convex hull of a bounded point-set S in a real Hilbert space E. This note is concerned with the following questions, which arise for example [1] in connection with the theory of "ϵ-convex sets."

Consider a point $p\epsilon K$. Is it true that every large enough closed ball, which contains p, must also meet S? How large is "large enough"?

It is readily seen that the answer to the first question is in general negative if p lies in the boundary ∂K of K (unless, of course, $p\epsilon S$). Thus it will be assumed throughout that p lies in the interior K° of $K=K(S)$.

With the notation

$$\delta=\text{diam } (S)=\text{diam } (K)<\infty,$$

$$\Delta=d(p, \partial K)\leq\delta,$$

define $R^*(p)$ and $R(p)$ by

$$R^*(p)=\delta^2/2\Delta,$$

$$R(p)=R^*(p)+\delta.$$

For $p\epsilon K^\circ$, with K, S, E, $R^*(p)$ and $R(p)$ as above, the following results will be proved:

THEOREM 1: *Every closed ball* B *with radius* $>$ R*(p) *for which* p$\epsilon\partial$B *meets* S.

THEOREM 2: *Every closed ball* B *with radius* $>$ R(p), *which contains* p, *meets* S.

2. Proof of Theorem 1

Let B be a closed ball of radius $r>R^*(p)$, such that $p\epsilon\partial B$. We wish to show that B meets S.

Let c be the center of B, and u the unit vector in the direction of $p-c$. Thus $p=c+ru$. Let H_1 be the closed hyperplane tangent to B at p, i.e.

$$H_1=p+\{y:(y, u)=0\}.$$

[1] J. Perkal, Sur les ensembles ε-convexes, Colloq. Math. 4 (1956), pp. 1–10.

Let E_1 be that one of the closed halfspaces of E determined by H_1 which contains c and thus contains B.

Since $r > R^*(p) = \delta^2/2\Delta$, we can choose $\eta < \Delta$ such that $r\eta > \delta^2/2$. Let H_2 be the hyperplane parallel to H_1 which passes through the point $q = c + (r-\eta)u$. Then H_2 is at distance η from H_1, and lies in E_1.

Let E_2 be that one of the closed halfspaces of E determined by H_2 which does not contain H_1. Then $E_2 - H_2$ is met by the open ball of radius Δ centered at p, e.g. at the midpoint of q and $c + (r-\Delta)u$. This ball lies wholly in K, so $E_2 - H_2$ meets K. Thus the closed convex set $E - (E_2 - H_2)$ does not contain all of the convex hull $K = K(S)$, and so does not contain all of S. Hence $E_2 \cap S$ is nonempty; to complete the proof of theorem 1, it suffices to prove that $E_2 \cap S \subset B \cap S$.

Consider then any $x \in E_2 \cap S$. We need to show that

$$d(x, c) = \|x - c\| \leq r. \tag{2.1}$$

Note that $x - c$ admits a unique orthogonal decomposition relative to u, i.e.,

$$x - c = y + \theta u, \qquad (y, u) = 0. \tag{2.2}$$

This yields

$$||x - c||^2 = ||y||^2 + \theta^2. \tag{2.3}$$

Because $x \in E_2$, we have

$$\theta \leq r - \eta. \tag{2.4}$$

Because $p \in K$ and $x \in S \subset K$, we have

$$\begin{aligned} \delta^2 = [\operatorname{diam}(K)]^2 &\geq \|x - p\|^2 \\ &= \|(x-c) - (p-c)\|^2 = \|y - (r-\theta)u\|^2 \\ &= \|y\|^2 + (r-\theta)^2 = \|y\|^2 + \theta^2 + r^2 - 2r\theta. \end{aligned}$$

Combining this with (2.3) gives

$$\|x - c\|^2 \leq \delta^2 + 2r\theta - r^2,$$

and application of (2.4) then yields

$$\|x - c\|^2 \leq \delta^2 + 2r(r-\eta) - r^2 = r^2 - (2r\eta - \delta^2).$$

Since η was so chosen that $2r\eta > \delta^2$, (2.1) is proved.

3. Proof of Theorem 2

We shall show that theorem 2 follows from theorem 1. Let B be a closed ball of radius $r - R(p)$ such that $p \in B$. There are two cases to be considered, depending on the distance of p from the center c of B.

If $d(c, p) \leq R^*(p) = R(p) - \delta$, then B contains the closed ball B' of radius δ centered at p. Since

$$\delta = \operatorname{diam}(S) = \operatorname{diam}(K),$$

while $p \in K$, it follows that $K \subset B' \subset B$ and thus that $S \subset B$. So B certainly meets S.

Now suppose $d(c, p) > R^*(p)$. Let B^* be the closed ball of radius $d(c, p)$ centered at c. Then B^* has radius $> R^*(p)$, and $p \in \partial B^*$. By theorem 1, B^* meets S. Since $B^* \subset B$, B meets S.

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The Partitioning of Interaction in Analysis of Variance

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A method is presented for the analysis of data representing functions of two variables, when the response can be tabulated in a rectangular array. The procedure is based on a partitioning of the row by column interaction effects into a sum of terms, each of which is the product of a row factor by a column factor. The factors in each term are estimated by a method involving the extraction of characteristic roots.

The method contains as special cases a number of procedures used for the handling of non-additivity in two way arrays. It is very useful for the fitting of empirical surfaces, but is also applicable to cases in which the data depend on qualitative rather than quantitative factors.

Comparisons with other techniques are made and an illustrative example is given.

Key words: Factorial Experiments; interaction; nonadditivity; principal components; surface fitting.

1. Introduction

In a previous paper [8],[1] a method was presented for the analysis of data representing functions of two variables, when the response can be tabulated in a rectangular array. The analysis was based on the assumption of a "linear model," the validity of which was subject to verification in each individual case. Essentially, the linear model assumption is that the elements of each row, when plotted against the column averages of the table (or that the elements of each column, when plotted against the row averages), provide a straight line. It was pointed out in [8] that when the linear model assumption does not hold, the analysis can be handled in two ways. The first of these simply generalizes the assumption of "linearity," required by the linear model, to one of quadratic behavior, as discussed in reference [9].

The second way of dealing with more complex models than are covered by the linearity assumption is to attack the problem of row by column interaction in a completely systematic way, by partitioning this term in as many individual terms as are required by the data. Thus, no prior assumptions such as linearity, or concurrence [7], or quadratic behavior [9] are made; to a considerable extent the data themselves generate the model. It is this approach that we wish to discuss in the present paper.

We will assume that the response is a quantitative variable, generally a measure of some property of a material or of a system. The nature of the data, and of the problem behind the data then depends on the nature of the two independent variables (which are represented by the rows and the columns of the two-way table). These may be qualitative or quantitative, or mixtures of both types, and the interpretation of the analysis will depend on which of these situations pertains. For the sake of brevity, we will describe the application of the method to only one illustrative example. Other interesting applications of the method will be presented in subsequent papers.

[1] Figures in brackets indicate the literature references at the end of this paper.

A basic feature of the method discussed in the previous publication [8] was the division of the analysis in two phases, the first of which was referred to as the analysis for "internal structure." This first phase led to an expression relating the response to sets of parameters, each of which was a function of only one variable. The second phase of the analysis consisted in relating these parameters to the original row and column variables. The present analysis, while far more general, retains this basic feature.

In the course of our discussion we will call attention to the relation between the proposed approach and other models for two-way data. We will also discuss the similarities, as well as the differences, between this approach and some related procedures discussed in the literature. In a broad sense, all these procedures, including the one presented in this paper, derive from the "method of principal components," the development of which is due primarily to Harold Hotelling [4]. Thus, while the method presented in this paper rests largely on well-known mathematical results, the approach is nevertheless novel. In particular, while practically all discussions of this method of principal components are nonsymmetrical with respect to the rows and the columns of the two-way table, the present approach treats the rows and the columns in the same identical way.

Ideas similar to those presented here have been discussed by E. J. Williams [17] and by Pike and Silverberg [11].[2] Williams makes explicit use of the theory of latent roots underlying the method of principal components, but approaches the data analysis problem from a somewhat different point of view. Pike and Silverberg make no use of the method of principal components.

While this paper was being prepared for publication, a most interesting paper by Gollob [3] was published. It is apparent that we developed the same basic model independently of each other.

A basic difference between Gollob's method and mine lies in what I believe to be a novel approach to the question of degrees of freedom in principal component analysis. This approach is discussed in detail in the present paper.

2. The Analysis of Internal Structure

Let x_i and y_j denote the levels of the two independent variables or categories, represented by the rows and columns of the table and z_{ij} the observed value of the measured (dependent) variable for x_i and y_j.

Assume that the table consists of m rows and n columns. We begin by writing the model usually adopted in analysis of variance:

$$z_{ij}=\mu+\rho_i+\gamma_j+\eta_{ij} \qquad \begin{cases} i=1 \text{ to } m \\ j=1 \text{ to } n \end{cases}. \tag{1}$$

Here, μ is a constant, estimated by the grand-mean; ρ_i is the "row main effect" and γ_j the "column main effect"; the quantity η_{ij} represents the "interaction between row i and column j."

As usual we impose the conditions:

$$\sum_i \rho_i=0 \qquad \sum_j \gamma_j=0 \qquad \sum_i \eta_{ij}=\sum_j \eta_{ij}=0. \tag{2}$$

It should be noted that eq (1) expresses the function of two variables, z_{ij}, *partly* in terms of two functions of a single variable each, ρ_i and γ_j, but that it also involves a new function of two variables, η_{ij}.

It is often assumed that η_{ij} is a random variable, of mean zero, and standard deviation σ. In that case, which is known as the "additive case," eq (1) constitutes a real simplification: the replacement of a function of two variables by two functions of one variable each.

[2] For calling my attention to the latter reference I am indebted to Professor David L. Wallace.

When the assumption of additivity is not definitely known to be valid, the analysis can proceed only on the basis of some other definite assumption concerning the structure of η_{ij}. The assumption we will adopt in this paper is expressed by the following equation:

$$\eta_{ij} = \theta u_i v_j + \theta' u_i' v_j' + \theta'' \mu_i'' v_j'' \ldots + \epsilon_{ij} \tag{3}$$

where ϵ_{ij} is a random variable of zero mean and standard deviation σ. When η_{ij} is expressed by only a few terms of the multiplicative type $\theta u_i v_j$, eq (3) also constitutes a real simplification. Without the loss of generality, we may impose the constraints:

$$\sum_i u_i = \sum_i u_i' = \ldots = \sum_j v_j = \sum_j v_j' = \ldots = 0 \tag{4}$$

$$\sum_i u_i^2 = \sum_i u_i'^2 = \ldots = \sum_j v_j^2 = \sum_j v_j'^2 = \ldots = 1. \tag{5}$$

The problem is to find estimates for the new parameters $\theta, \theta', \ldots u, u', \ldots, v, v', \ldots$ and for the standard deviation of random error, σ. Equations (1) and (3) together with the constraints expressed by eqs (4) and (5), constitute our model. We will refer to the quantities μ, ρ_i, γ_j, θ, u_i, v_j, θ', u_i', v_j', etc. as "structural parameters."

3. Solution by Least Squares [3]

The interaction η_{ij} is estimated by the residual

$$d_{ij} = z_{ij} - \hat{\mu} - \hat{\rho}_i - \hat{\gamma}_j \tag{6}$$

where $\hat{\mu}$, $\hat{\rho}_i$, $\hat{\gamma}_j$ are the usual estimates

$$\hat{\mu} = z_{..}; \quad \hat{\rho}_i = z_{i.} - z_{..}; \quad \hat{\gamma}_j = z_{.j} - z_{..} \tag{7}$$

The dot notation indicates here, as usual, averaging over the subscript replaced by the dot. To fit the interaction, consider first the model:

$$\eta_{ij} = \theta u_i v_j + \epsilon_{ij} \tag{8}$$

subject to the constraints expressed by eqs (4) and (5). We obtain estimates for θ, u_i, and v_j by minimizing with respect to θ, u_i, and v_j, the quantity:

$$S = \sum_i \sum_j (d_{ij} - \theta u_i v_j)^2 - \lambda_1(\Sigma u^2 - 1) - \lambda_2(\Sigma v^2 - 1) - 2\mu_1 \Sigma u_i - 2\mu_2 \Sigma v_j \tag{9}$$

where the λ and μ's are Lagrange multipliers. Carrying out the calculations, one obtains the relations:

$$u_k = \theta^{-1} \sum_j d_{kj} v_j \tag{10}$$

$$v_l = \theta^{-1} \sum_i d_{il} u_i \tag{11}$$

[3] The derivation in this section is essentially the same as that given by Williams [17].

Eliminating v_j between (10) and (11), and setting

$$s_{ik} = \sum_j d_{ij} d_{kj} \tag{12}$$

one obtains:

$$\theta^2 u_k = \sum_i u_i s_{ik}. \tag{13}$$

In matrix notation, eqs (12) and (13) become:

$$(s) = (d)(d)^T \tag{14}$$

and

$$\theta^2 (u) = (s)(u) \tag{15}$$

where T indicates a transpose, (s) is an $m \times m$ matrix, and (u) is an $m \times 1$ vector.

It follows from (15) that θ^2 is an eigenvalue of the matrix (s) and that (u) is an associated eigenvector.

Similarly it can be shown that:

$$\theta^2 (v) = [(d)^T (d)](v). \tag{16}$$

where (v) is an $n \times 1$ vector.

The desired minimization of S (eq (9)) is accomplished by taking the *largest* eigenvalue θ^2 of the matrix (s). Furthermore, an important consequence, which can readily be proved,[4] is that if θ^2 is chosen to be the *largest* eigenvalue of the matrix (s), and one attempts to find the values of θ', u_i' and v_j' such that:

$$\sum \sum_j [(d_{ij} - \theta u_i v_j) - \theta' u_i' v_j']^2 \tag{17}$$

is a minimum, subject to the constraints (4) and (5), it turns out that θ'^2 is simply the second-largest eigenvalue of the same matrix (s), and (u') the associated eigenvector, with a similar situation for (v').

Thus, by obtaining the *complete* set of eigenvalues of (s), and the associated sets of vectors (u), (v), (u'), (v'), etc., one actually obtains the least-squares solution of *all* parameters in eq (3).

4. An Analog to Analysis of Variance

In terms of sample estimates, eq (3) leads to the equation:

$$d_{ij} = \hat{\theta} \hat{u}_i \hat{v}_j + \hat{\theta}' \hat{u}_j' \hat{v}_j' + \ldots + e_{ij}. \tag{18}$$

The rank of the (s) matrix satisfies the inequality:

$$\text{rank } (s) \leq \min\ (m, n) - 1. \tag{19}$$

Consequently, the number of terms on the right side of (18) is at most min $(m, n) - 1$. In analogy to the usual interpretation of the results of analysis of variance, a decision will first be made (see following section) on how many terms of the type $\theta u_i v_j$ should be retained in the model; the residual sum of squares is then used for an estimation of the variance of ϵ.

[4] See, for example: Harman, H. H., Modern Factor Analysis (The University of Chicago Press, Chicago, 1960, chap. 9).

Assume that we calculate *all* terms of the type $\theta u_i v_j$ so that ϵ_{ij} is taken equal to zero. Then we obtain at once:

$$\sum_i \sum_j d_{ij}^2 = \hat{\theta}^2 + \hat{\theta}'^2 + \hat{\theta}''^2 + \ldots . \tag{20}$$

This follows from eq (18) and the orthogonality property of eigenvectors, by which all cross-products are zero.

Equation (20) constitutes an actual partitioning of the sum of squares of interaction. One is therefore tempted to try an approach of the analysis of variance type. However, the θ^2 are *not* quadratic forms in the original measurements z_{ij}. Nevertheless, it is possible to formulate the problem in analysis of variance language, by virtue of the following considerations.

Let $\mathcal{N}(\mu, \sigma)$ represent a normal population of mean μ and standard deviation σ.

Suppose first that the z_{ij} are a random sample from a normal population $\mathcal{N}(0, 1)$. Then, the quantities $\hat{\theta}^2, \hat{\theta}'^2, \hat{\theta}''^2$, etc., each form a definite statistical population. Let:

$$M_1 = E(\hat{\theta}^2)_{\mathcal{N}}, \; M_2 = E(\hat{\theta}'^2)_{\mathcal{N}}, \; M_3 = E(\hat{\theta}''^2)_{\mathcal{N}}, \ldots \tag{21}$$

where E represents an expected value and the symbol $\mathcal{N}$ refers to the standard normal distribution $\mathcal{N}(0, 1)$.

If the normal population in question had a variance of σ^2, instead of unity, the quantities M_1, M_2, M_3, etc., would simply be multiplied by σ^2.

Therefore, the ratios of the $\hat{\theta}^2$ values obtained from a random sample from $\mathcal{N}(0, \sigma)$ to the corresponding M values are *all* estimates of σ^2.

If, now, a matrix of data z_{ij} is given, and if the interaction terms are nothing but random gaussian error, then the ratios of the $\hat{\theta}^2$ obtained from these data to the corresponding M-values obtained from an $\mathcal{N}(0, 1)$ matrix of the same dimensions are simply estimates of σ^2.

Thus, the M-values fulfill a role that is quite analogous to that of the degrees of freedom in ordinary analysis of variance. More specifically, the M are such that, for random gaussian interaction, the ratio of each $\hat{\theta}^2$ by the corresponding M is an unbiased estimate of σ^2.

Carrying the above argument a little further, and using heuristic reasoning, we may expect that if the real model contains say, k terms of the type $\theta u_i v_j$, then the corresponding k values of $\hat{\theta}^2$ will be inflated by the systematic effects of these terms, while the *remaining* terms, say:

$$\hat{\theta}_{k+1}^2, \hat{\theta}_{k+2}^2, \text{etc.}$$

will only be estimates of

$$M_{k+1}\sigma^2, \; M_{k+2}\sigma^2, \text{ etc.}$$

Thus:

$$\theta_{k+1}^2/M_{k+1}, \; \theta_{k+2}^2/M_{k+2}, \text{ etc.},$$

will all be estimates of σ^2.

In this way a judgment can be made as to the number of terms, k, that should be retained in the model, as will be shown in the discussion of the example, further in this paper.

5. A Monte-Carlo Study

To verify the above assumptions, and obtain reliable numerical estimates for the M-values, a series of Monte Carlo experiments were carried out, as follows.

For specific values of m and n, matrices containing random normal deviates from $\mathcal{N}(0, 1)$, were analyzed by the method outlined above. For each m and n combination, 625 such matrices were analyzed, and the estimates of the corresponding M-values computed. The averages of the θ^2, for 625 sets, and their standard deviations are listed in tables A1 and A2, respectively, of the appendix.

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In an $m \times n$ matrix, the number of degrees of freedom for interaction is $(m-1)\ (n-1)$. If $\sigma^2=1$, the expected value of the sum of squares for interaction is also $(m-1)\ (n-1)$. Thus, the expected value of the sum of the M-values, for any given matrix of size $m \times n$, is $(m-1)\ (n-1)$. In view of this fact, it seemed advantageous to compute the ratio of each value to $(m-1)\ (n-1)$. These ratios, expressed as percent, are denoted "Percent of Total Interaction Degrees of Freedom," and are listed in table A3. They were obtained by a weighted least squares adjustment process, using the reciprocals of the squares of the observed standard deviations as relative weights, and introducing the constraint that the sum of the percentages be 100. The values in table A3 lend themselves to more precise interpolation than those of table A1.

Additional Monte Carlo studies were carried out to verify the conjecture that if one or more terms of the type $\theta u_i v_j$ were actually present in the true model, the *subsequent* terms would not be affected by these systematic terms. The experiments showed that the introduction of even very large systematic terms of the type $\theta u_i v_j$ had the effect of only slightly inflating the subsequent terms; these remained, for all practical purposes, acceptable estimates of the experimental error.

6. Practical Use of the Method

Let z_{ij} be given in the form of an $m \times n$ matrix. We assume that z_{ij} is a function of x and y, where the values of x correspond to the rows and the values of y to the columns of the matrix.

We first fit a model of the type:

$$z_{ij}=\mu+\rho_i+\gamma_j+\eta_{ij}=\mu+\rho_i+\gamma_j+\theta u_i v_j+\theta' u_j' v_j'+ \ . \ . \ .$$

to the data, ignoring momentarily the numerical values, if any, of x_i and y_j.

The usual analysis of variance yields the estimates:

$$\hat{\mu}=z_{..};\ \hat{\rho}_i=z_{i.}-z_{..};\qquad \hat{\gamma}_j=z_{.j}-z_{..}$$

and the residuals:

$$\hat{\eta}_{ij} \equiv d_{ij}=z_{ij}-\hat{\mu}-\hat{\rho}_i-\hat{\gamma}_j=z_{ij}+z_{..}-z_{i.}-z_{.j}.$$

For purposes which will become apparent in the discussion of the illustrative example, we "standardize" the parameter estimates $\hat{\rho}_i$ and $\hat{\gamma}_j$, by dividing each of them by the square root of the corresponding sum of squares. Thus, writing:

$$R=\sqrt{\sum_i \hat{\rho}_i^2},\qquad G=\sqrt{\sum_j \hat{\gamma}_j^2}$$

$$r_i=\frac{1}{R}\hat{\rho}_i,\qquad c_j=\frac{1}{G}\hat{\gamma}_j \tag{22}$$

our model becomes:

$$z_{ij}=\mu+Rr_i+Gc_j+\theta u_i v_j+\theta' u_i' v_j'+ \ . \ . \ . \ . \tag{23}$$

Applying a diagonalization technique to the matrix $(d) \cdot (d)^T$, we find:

(a) a set of eigenvalues

$$\hat{\theta}^2,\ \hat{\theta}'^2,\ \hat{\theta}''^2,\ . \ . \ .$$

(b) a set of u vectors, each of dimensions $m \times 1$:

$$(u),\ (u'),\ (u''),\ . \ . \ .$$

(c) a set of v vectors, each of dimensions $n \times 1$:

$$(v),\ (v'),\ (v''),\ . \ . \ .$$

We now tabulate the results as follows (tables 1 and 2):

TABLE 1. *Row-dependent parameters*

Row variable	Row effects, r_i	u_i	u_i'	u_i'' . . .
x_1	r_1	u_1	u_1'	u_1'' . . .
x_2	r_2	u_2	u_2'	u_2'' . . .
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
x_m	r_m	u_m	u_m'	u_m'' . . .

TABLE 2. *Column-dependent parameters*

Column varaible	Column effects, c_j	v_j	v_j'	v_j'' . . .
y_1	c_1	v_1	v_1'	v_1'' . . .
y_2	c_2	v_2	v_2'	v_2'' . . .
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
y_n	c_n	v_n	v_n'	v_n'' . . .

We form the analysis of variance table (table 3):

TABLE 3. *Analysis of variance* [a]

Source	*DF*	*SS*	$MS=\frac{SS}{DF}$
Total	$m\cdot n$	$\sum_i\sum_j z_{ij}^2$	
μ	1	$mn\mu^2$	
ρ_i	$m-1$	nR^2	$nR^2/(m-1)$
γ_j	$n-1$	mC^2	$mC^2/(n-1)$
η_{ij}	$(m-1)(n-1)$	$\sum_i\sum_j d_{ij}^2$	$\sum\sum d_{ij}^2/(m-1)(n-1)$
$\theta u_i v_j$	M_1	$\hat{\theta}^2$	$\hat{\theta}^2/M_1$
$\theta' u_i' v_j'$	M_2	$\hat{\theta}'^2$	$\hat{\theta}'^2/M_2$
$\theta'' u_i'' v_j''$	M_3	$\hat{\theta}''^2$	$\hat{\theta}''^2/M_3$
.	.	.	.
.	.	.	.
.	.	.	.

[a] The term "analysis of variance" is used here in a generalized sense: as in classical analysis of variance, the sum of squares is partitioned and mean squares are calculated. Under the null-hypothesis of "no effect," the mean squares are unbiased estimates of the error variance. The *M*-values are "degrees of freedom" only in the sense of appropriate divisors for the corresponding sums of squares, as explained in the text.

We now examine the mean squares corresponding to the breakdown of the η_{ij} into the sum of multiplicative terms $\theta u_i v_j$, $\theta' u_i' v_j'$, etc. At the present time, no distribution theory is available for these mean squares, and exact tests of significance cannot be carried out. Nevertheless, even an intuitive appraisal of the mean squares generally leads to fairly clear-cut decisions concerning the number of product terms that should be retained in the model. This will be illustrated in the next section, which deals with an illustrative example.

7. Illustrative Example

The data in table 4 were obtained in an experiment designed to measure the absorption of gamma radiation by lead (Pb) and by aluminum (Al).

For each of the two metals, two sets of measurements were made, each of which involved 5 different values for the distance between the radioactive source and the counting instrument. The thickness of the absorbing metal was varied by stacking plates of identical thickness upon each other, and placing these stacks between the source and the counter. Five thicknesses were used, obtained by making stacks of respectively 1, 3, 6, 7, and 10 plates. The response is the logarithm, to the base 10, of the number of pulses per second. The thickness of a single plate was not the same for the two metals, but this does not invalidate our analysis.

TABLE 4. *Absorption of gamma radiation by lead and aluminum*[a]

Row, i	Distance	Set	Number of plates, k_j				
	cm		1	3	6	7	10
			Lead				
1	3.8		1.801	1.765	1.696	1.670	1.606
2	5.2		1.621	1.572	1.516	1.486	1.425
3	6.0	I	1.526	1.481	1.406	1.401	1.333
4	9.0		1.222	1.169	1.102	1.078	1.010
5	12.5		0.973	0.939	0.862	0.850	0.781
6	3.8		1.805	1.768	1.704	1.680	1.615
7	5.2		1.609	1.572	1.511	1.482	1.408
8	6.0		1.494	1.461	1.387	1.324	1.315
9	9.0	II	1.233	1.208	1.130	1.111	1.046
10	12.5		0.978	0.930	0.870	0.844	0.779
			Aluminum				
11	3.8		1.834	1.818	1.811	1.790	1.777
12	5.2		1.632	1.613	1.600	1.603	1.597
13	6.0	I	1.509	1.482	1.476	1.454	1.447
14	9.0		1.249	1.224	1.204	1.211	1.179
15	12.5		0.986	0.971	0.966	0.960	0.943
16	3.8		1.916	1.913	1.884	1.887	1.871
17	5.2		1.732	1.723	1.698	1.696	1.674
18	6.0	II	1.632	1.624	1.592	1.588	1.579
19	9.0		1.344	1.341	1.312	1.311	1.290
20	12.5		1.118	1.118	1.106	1.086	1.066

[a] Tabulated value $= \log_{10} \frac{\text{pulses}}{\text{sec}}$ (corrected for background noise).

The first phase of the analysis was carried out on a two-way table, in which the rows represent combinations of three factors: metal, distance, and set. The columns of the table correspond to the 5 levels of "number of plates," k_j.

Using the notation introduced previously in this paper, the analysis of variance is given in table 5, where the M values given in table A1 were used for the degrees of freedom. The structural parameters are given in table 6.

TABLE 5. *Absorption of Gamma Radiation, Analysis of Variance*[a]

Source	*DF*	*SS*	*MS*	
Total	100	205.34167	.....	
μ	1	195.97757	.....	
ρ_i	19	9.106487	0.4793	
γ_j	4	0.190188	.0475	
η_{ij}	76	.067447	.000887	
$\theta u_i v_j$	31.9	.063801	.002000	
$\theta' u_i' v_j'$	21.3	.002083	.000098	
$\theta'' u_i'' v_j''$	14.3	.000847	.000059	0.000083
Residual[b]	8.5	.000716	.000084	

[a] See footnote a of table 3.

[b] The total number of multiplicative terms for these data is 4; hence the residual is identical with the fourth eigenvalue. The degrees of freedom for the residual are obtained by difference.

TABLE 6. *Absorption of Gamma Radiation, Structural Parameters*

Model: $z_{ij} = 1.40 + 1.3495 r_i + 0.0975 c_j + 0.2526 u_i v_j + \epsilon_{ij}$

Distance	Set	r_i	u_i		k_j	c_j	v_j
3.8		0.2280	0.2258		1	−0.6447	−0.6487
5.2		.0919	.2135				
6.0	Pb I	.0218	.2061		3	−.2494	−.2540
9.0		−.2102	.2623				
12.5		−.3845	.2148		6	−.0848	−.0767
3.8		.2330	.2043		7	.3556	.3689
5.2		.0863	.2336				
6.0	Pb II	−.0028	.2290		10	.6233	.6105
9.0		−.1884	.2117				
12.5		−.3851	.2225				
3.8		.3009	−.2117				
5.2		.1549	−.2850				
6.0	Al I	.0546	−.2009				
9.0		−.1382	−.1908				
12.5		−.3221	−.2652				
3.8		.3662	−.2359				
5.2		.2258	−.2036				
6.0	Al II	.1505	−.2052				
9.0		−.0595	−.2088				
12.5		−.2231	−.2166				

Since the mean squares in table 5 are small for the second, third, and fourth multiplicative terms, when compared to that of the first multiplicative term, the analysis of variance indicates quite conclusively the need for a single multiplicative term, in addition to the usual additive terms (grand average, row main effects and column main effects).

The model emerging from this analysis is given by the following equation:

$$z_{ij} = 1.400 + 1.3495 r_i + 0.0975 c_j + 0.2526 u_i v_j + \epsilon_{ij}$$

where r_i, u_i, c_j, and v_j are listed in table 6, and ϵ_{ij} is a random error, with standard deviation $\hat{\sigma} = 0.0091$.

This model may be simplified by examining the structural parameters. We first note the great similarity between v_j and c_j.[5]
Assuming that $v_j = c_j$, our model becomes

$$z_{ij} = 1.400 + 1.3495 r_i + (0.0975 + 0.2526 u_i) v_j + \epsilon_{ij} \tag{24}$$

Our next task is to study the structure of the parameters r_i and u_i in terms of the variables of which they are functions, i.e., distance and set. For the parameter u_i, a formal analysis is hardly necessary. It is evident from table 6 that u_i has essentially the same absolute value for all 20 combinations of distance with set, and that its sign is + for Pb and − for Al. Since $\sum_i u_i^2 = 1$. we obtain:

$$u_i = \pm \sqrt{\frac{1}{20}} = \pm 0.2236 \tag{25}$$

the + sign applying to Pb, and the − sign to Al.

To study r_i, we make a two-way analysis of the same type as above, considering r_i as the response, and identifying the rows with distance and the columns with the four sets (Pb I, Pb II, Al I, Al II). The results are shown in table 7a. From the analysis of variance it is clear that a simple additive model is appropriate. Thus, we write the model

$$r_i = A_d + B_s \tag{26}$$

with the values of the new parameters A_d and B_s listed in table 7b. Combining eqs (24), (25) and (26) yields:

$$z_{ij} = 1.400 + 1.3495 (A_d + B_s) + (0.0975 \pm 0.0565) v_j + \epsilon_{ij} \tag{27}$$

where the + sign applies to Pb and the − sign to Al.

TABLE 7a. *Analysis of variance of* r_i

Source	*DF*	*SS*	*MS*
Total	20	1.0000	0.05
Mean	1	0	0
Distance	4	0.9300	0.2325
Set	3	.0688	.0229
Distance × set	12	.001203	.000100
EV1 [a]	8.4	.000752	.000090
EV2	3.0	.000343	.000114
EV3	0.6	.000108	.000180

[a] EV1, EV2, and EV3 denote the three eigenvalues into which the interaction Distance × Set is partitoned.

[5] Here the advantage of "standardizing" the $\hat{\rho}_i$ and $\hat{\gamma}_j$ into r_i and c_j becomes apparent. An identical relationship between v_j and c_j (or between v_j and $(-c_j)$) corresponds to a proportional relationship between v_j and γ_j. Such a proportional relationship might have escaped attention, unless the values had been plotted on a graph.

TABLE 7b. *Values of parameters in equation:* $r_i = A_d + B_s$

Distance	A_d		Set	B_s
3.8	0.2820		Pb I	−0.0506
5.2	.1397		Pb II	−.0514
6.0	.0560		.....	.....
9.0	−.1491		Al I	.0100
12.5	−.3287		Al II	.0920

All that remains to be done is to fit appropriate curves to the parameter A_d, B_s, v_j as functions of the variables distance, metal, and k_j (number of plates) respectively, using the values listed in table 7b (for A_d and B_s) and in table 6 (for v_j).

In view of the near-identity of the two values of B_s for Pb, both sets for this metal can be expressed by the same equation. Equation (27) then leads to the following model equations:

For Pb (sets I and II):

$$z_{ij} = 1.331 + 1.3495A_d + 0.1540v_j + \epsilon_{ij}. \tag{28}$$

For Al:

set I: $z_{ij} = 1.414 + 1.3495A_d + 0.0410v_j + \epsilon_{ij}$ (29)

set II: $z_{ij} = 1.524 + 1.3495A_d + 0.0410v_j + \epsilon_{ij}$. (30)

A plot of A_d versus distance and of v_j versus number of plates would show smooth relationships for both these parameters. In fact, the plot of v_j versus the number of plates is simply a straight line, which is in accordance with the theory of absorption of radiation.

We need not complete this phase of the analysis, since it involves no novel features.

Factorial types of data, such as those used in our illustration (table 4) are often analyzed by the conventional analysis of variance, in which the sum of squares is partitioned into main effects and interactions. To interpret such an analysis, it must be ascertained whether the interactions that are found to be significant depend in a systematic way on the factors. When this is the case, the model is not additive, and then the nature of the interactions must be further elucidated. This is precisely what the method of analysis proposed in this paper is intended to do.

Thus, in our analysis of the illustrative example, in which the three factors "type of metal," "distance," and "sets" were merged into one (the "rows" of the table), a clear dichotomy was nevertheless indicated by the parameter u_i, which sharply differentiated between Pb and Al, and led to the entirely different coefficient of v_j in eq (28) as compared to eqs (29) and (30). This is of course an "interaction" between "type of metal" and "number of plates," but the advantage of our approach is that rather than merely indicating the *presence* of an interaction, the parameter leads to a quantitative expression for it.

8. Relationship to Models not Based on Principal Component Analysis

a. The Additive Model

Clearly, (23) expresses an additive model if and only if all multiplicative terms vanish (or become mere random experimental error). An additive model is therefore diagnosed if none of the mean squares obtained in the partitioning of the interaction are large with respect to subsequent mean squares.

b. The Linear Model

This model [7, 8] is given by the equation:

$$z_{ij} = \mu + \rho_i + \gamma_j + (\beta_i - 1)\gamma_j + \epsilon_{ij}$$

where $\bar{\beta}=1$.

It can be written as:

$$z_{ij}=\mu+Rr_i+Gc_j+(\beta_i-1)Gc_j+\epsilon_{ij}.$$

In order that this model apply, it is necessary that the following two conditions be fulfilled:

1. The partitioning of the interaction must yield only one significant product term;
2. We must have, either

$$v_j=c_j \qquad (\text{all } j)$$

or

$$v_j=-c_j \qquad (\text{all } j).$$

Obviously a linear model also holds when

$$z_{ij}=\mu+\rho_i+\gamma_j+\rho_i(\delta_j-1)+\epsilon_{ij}; \qquad \bar{\delta}=1.$$

In this case, condition 2 above becomes:

$$u_i=r_i \qquad (\text{all } i)$$

or

$$u_i=-r_i \qquad (\text{all } i)$$

c. The Concurrent Model

This model [7, 8] is a special case of b.

It is represented by the relation:

$$z_{ij}=\mu+\rho_i+\gamma_j+K\rho_i\gamma_j+\epsilon_{ij}.$$

A set of data will be represented by a concurrent model if and only if the following three conditions are simultaneously fulfilled:

1. Only one significant product term results from the partitioning of the interaction.

2. $u_i=r_i$ or $u_i=-r_i$ (all i)
3. $v_j=c_j$ or $v_j=-c_j$ (all j).

d. Tukey's One Degree of Freedom for Non-Additivity

In 1949, Tukey [13] proposed a test for nonadditivity consisting in the extraction of one degree of freedom from the row by column interaction. This test can be interpreted in terms of the model

$$z_{ij}=\mu+\rho_i+\gamma_j+K\rho_i\gamma_j+\epsilon_{ij}$$

which is identical with what we have called the concurrent model.

In Tukey's procedure, the sum of squares, with one degree of freedom, for $K\rho_i\gamma_j$, is tested against the remaining sum of squares, with $[(m-1)\ (n-1)-1]$ degrees of freedom.

We can regard this model as a special case of the linear model (case b).

If we write:

$$z_{ij}=\mu+\rho_i+\gamma_j+K\rho_i\gamma_j+[(\beta_i-1)-K\rho_i]\gamma_j+\epsilon_{ij}$$

the allocation of degrees of freedom is 1 for $K\rho_i\gamma_j$, and $(m-2)$ for $[(\beta_i-1)-K\rho_i]\gamma_j$, provided that β_i and K are estimated by the appropriate procedure [7]. Thus a valid test for $K\rho_i\gamma_j$ is obtained by computing the F ratio of the mean squares corresponding to these two terms, with 1 and $(m-2)$ degrees of freedom. Tukey's procedure consists in testing the term $K\rho_i\gamma_j$ versus the combined

mean square for $[(\beta_i-1)-K\rho_i]\gamma_j$ *and* ϵ_{ij}. The denominator now has $[(m-1)(n-1)-1]$ degrees of freedom. Thus carried out, the test overlooks the possible significance of the term $[(\beta_i-1)-K\rho_i]\gamma_j$. It appears preferable to consider first the linear model, and separate out one degree of freedom for concurrence. Indeed, Tukey's test will tend to lead to an erroneous conclusion of additivity everytime the model is linear without being concurrent.

e. The Vacuum-Cleaner Model

In a paper appearing in 1962, Tukey [14] presented what may be considered as a repeated twofold application of the linear model, both row-wise and column-wise. He referred to it as the "vacuum cleaner model." The first "sweep" of the vacuum cleaner model may be represented by the following equation:

$$z_{ij}=\mu+\rho_i+\gamma_j+K\rho_i\gamma_j+\beta_i\gamma_j+\rho_i\delta_j+\epsilon_{ij}$$

where

$$\sum_i \beta_i=\sum_j \delta_j=0$$

and

$$\sum_i \beta_i\rho_i=\sum_j \gamma_j\delta_j=0.$$

If data obeying this model are analyzed by the method of this paper, the first three eigenvalues will tend to extract all the information contained in the terms $K\rho_i\gamma_j$, $\beta_i\gamma_j$ and $\rho_i\delta_j$, but there will be no one-to-one-correspondence between the eigenvalues and these terms.

A characteristic feature of the vacuum cleaner model is that the "carriers" of each sweep are essentially the "coefficients" of the preceding sweep. For example, in the equation above, the three terms into which the interaction is partitioned are all linear functions of ρ_i and γ_j, the coefficients of the additive components. If an additional sweep were required, its term would be linear functions of β_i and δ_j, and so forth.

By contrast, the terms $\theta u_i v_j$ introduced consecutively in our method of analysis, are not made dependent upon each other. At each step, the *residuals alone* produce the new term $\theta u_i v_j$, whereas in the vacuum-cleaner model, the new terms are functions of the residuals *and* of the coefficients of the preceding sweep.

The preceding discussion shows that an analysis of two-way data carried out according to the method presented in this paper contains as special cases a number of methods not involving principal component analysis, and that application of this technique will allow us to recognize at once a number of frequently occurring special models.

9. Relationship to Other Approaches Involving Principal Component Analysis

As mentioned earlier in this paper, the idea of using principal component procedures in data analysis is not new. In the following we will show in what way the method presented in this paper differs from similar approaches.

The method of principal components originated in the field of psychology. We can visualize the basic problem in terms of a two-way table of m rows and n columns. Each row represents a different "subject" (e.g., a human being), and each column represents a particular psychological "test." The response is the score obtained by the ith "subject" in the jth "test." Here a vast conceptual difference exists between rows and columns. The object is to find a *small* number of linear combinations of the values in any particular row ("derived responses") that somehow typifies the overall response of the subject to the entire battery of tests. The model is formulated in terms of "characteristic vectors," equal in number to the "derived responses." For each derived response, the corresponding vector consists of the coefficients by which the n scores of any subject must be multiplied in order to obtain the linear combination yielding the derived response for that subject. The set of values obtained for all subjects for a particular derived response is denoted as the set of "scalar multipliers" corresponding to that vector.

In our model, no distinction in principle is made between vector components and scalar multipliers. They fulfill entirely analogous roles, one for the columns, and the other for the rows (they are the quantities denoted as v_j and u_i respectively). Furthermore, whereas in the classical approach, the calculation of principal components is generally carried out on the deviations of the original scores from the corresponding column means, we apply these calculations to the residuals from the additive model, i.e., after taking out both row and column means.

The literature dealing with the application of principal components to data analysis in the physical sciences contains some valuable contributions [1, 2, 3, 5, 6, 10, 12, 15, 17], but suffers largely from its adherence to the terminology and model building methods derived from the original field of application of this technique. In the first place, with the exception of reference [3], which will be discussed separately below, rows and columns are always treated as conceptually different entities. We have seen in our illustrative example that this distinction is unnecessary, even when the rows or columns represent discrete categories rather than controlled quantitative variables.

Secondly, one finds repeated reference to the "percentage of the variance accounted for by a particular latent root" [2, 5, 12]. While this may be a valid concept in psychological and similar applications, it could be a very misleading criterion in the analysis of physical and chemical data. Scientists in these fields generally base their acceptance of a residual sum of squares as an expression of experimental error on the agreement between the residual mean square and the variance of experimental error as known to them from previous experience. A good example is provided by Wernimont's data [15]. In analyzing jointly two spectrophotometers he obtains the two roots[6] (his Table III): $\lambda_1 = 7{,}896{,}525$ and $\lambda_2 = 2{,}553$, and a residual sum of squares of 151. The first root accounts for 99.9658 percent of the total sum of squares, and the second for only 0.0323 percent, yet Wernimont has shown that the second root is important and has given it a meaningful physical interpretation.

A third major difference between our approach and that of other authors (with the exception of [3]) is that we extract both components of the additive model (ρ_i and γ_j) *before* extracting characteristic roots. This leads, in the first place, to the possibility of treating rows and columns symmetrically, and, in the second place, to a set of completely "standardized" quantities (sum zero, sum of squares unity). As shown above, comparison of these quantities with the r_i and c_j allows for immediate recognition of important special cases.

The importance of this point can be seen from an examination of an illustrative example used by Simonds [12], and from his own analysis of these data. Starting with a 7×5 matrix of data, Simonds subjects the column average-corrected values to a principal component analysis, and decides that 2 vectors are sufficient to represent the data. His model is, accordingly:[7]

$$z_{ij} = \bar{z}_j + Y_{1,i} \cdot V_{1,j} + Y_{2,i} \cdot V_{2,j}.$$

The following values were obtained for the vectors $\bar{z}_j$ and $V_{1,j}$:

$$\bar{z}_j = [0.134 \quad 0.166 \quad 0.384 \quad 0.883 \quad 1.446]$$
$$V_{1,j} = [0.048612 \quad 0.066297 \quad 0.157030 \quad 0.401836 \quad 0.661341].$$

The vector $V_{1,j}$ is of course independent of the vector $\bar{z}_j$ since the latter had been removed from the data prior to the extraction of characteristic roots. Simonds fails to observe, however, that these two vectors are linearly related. If both vectors are normalized (by subtracting the mean and dividing by the square root of the sum of squares of deviation from the mean), one obtains:

$$\bar{z}_{j,\,\text{norm.}} = [-0.4195 \quad -0.3908 \quad -0.1957 \quad 0.2510 \quad 0.7550]$$
$$V_{1,j,\,\text{norm.}} = [\ -.4174 \quad -.3836 \quad -.2102 \quad .2577 \quad .7536].$$

[6] The first root given in Wernimont's table is actually 94,790,740. This value is in error; it should have been 7,896,525 [16].

[7] We follow as closely as possible his notation, modifying it only slightly, by introducing the subscripts i and j to represent rows and columns.

Thus, the model equation really contains one less vector than that given by Simonds. A finding of this type is of course important for the physical interpretation of data.

Similarly, Wernimont [15] takes no notice of the fact that, after normalization, the first characteristic vector he obtained in the analysis of single spectrophotometers, is identical with the vector of column averages, and that the corresponding vector of scalar multiples is identical with the vector of row averages. These facts, as seen above, indicate a concurrent model and this is precisely the model proposed by Wernimont. But an analysis in which both row and column averages have first been eliminated leads to a more exact testing procedure of the concurrence of the model.

Last, but not least, some of the pertinent papers that have come to the author's attention, including [3], suffer from a number of misconceptions regarding the proper number of degrees of freedom to be allocated to the successive eigenvalues. Others [2, 5, 12] ignore the matter altogether, by limiting their considerations to the "percentage of total variability explained" by the various eigenvalues. As pointed out earlier, the eigenvalues are genuine additive components of the interaction sum of squares, but they are *not* distributed as independent chi-square variates. Therefore the usual intuitive concepts concerning degrees of freedom do not apply. Our method is to define degrees of freedom as a quantity such that when the eigenvalue is divided by it, one obtains, for the case of random normal deviates of variance σ^2, an unbiased estimate of σ^2. In contrast, the following procedures are found in the literature. Morris and Morrissey [10], starting with a 42×31 matrix of original measurements, extract three roots, after initial subtraction of the column means. The residual sum of squares is then divided by 41×28, indicating that they regard as the proper formula for residual degrees of freedom, the expression $(m-1)\ (n-p)$, where p is the number of roots extracted. The formula seems to be based on the belief that the extraction of each root results in the loss of one degree of freedom for each of the $m-1$ independent rows. A similar formula is used by Wernimont [15]: here no adjustment (for the column-means) is made prior to the extraction of the roots, and the residual variance is calculated by dividing the residual sum of squares by $m(n-p)$. Judd et al. [6], appear to make no allowance, in terms of degrees of freedom for the fact that the eigenvalues are computed from the data. They state: "The variance for each set of data was computed in the usual way as the sum of squares of the differences between each input data and the corresponding value reconstituted from the mean and the first four characteristic vectors . . . divided by the number of input data." Gollob [3] defines mean squares corresponding to the partitioning of the interaction term on the basis of a heuristic argument. According to his definition, the "mean square" corresponding to the kth eigenvalue is the quotient of the eigenvalue by $(m+n-1-2k)$ (our notation).

A comparison was made between the values $m+n-1-2k$ and our empirical results. Table 8 lists both sets for matrices of various sizes. It is seen that the values $(m+n-1-2k)$ become less acceptable as m and n increase. In fact, whereas for a 4×4 matrix the value for $k=1$ is too low by about 21 percent, the corresponding value in a 16×16 matrix is too low by about 43 percent. The situation is reversed at the other end of the series of product terms (large k) where the estimate $(m+n-1-2k)$ is far too large. Thus, mean squares based on these "degrees of freedom" will be too large for the beginning terms and too small for the later terms. Comparisons of these "mean-squares" with an estimate of error obtained from within-cell replication will be vitiated by the biases inherent in these mean squares.

10. Computer Program

A computer program has been prepared by Mary N. Steel, for the application of the method of analysis presented in this paper, for data displayed in matrices of size up to 78×78. Details of this program, which is written in Fortran V, will be the subject of a separate publication.

The program includes, in addition to the analysis of variance for the additive effects, a partitioning of the row by column interaction into eigenvalues. It also tabulates the u and v vectors and the residuals after each successive step in the application of the method. Thus, residuals are given for the additive model, the model including one multiplicative term, two such terms, and so on.

TABLE 8. *Comparison of Gollob's formula for degrees of freedom with Monte-Carlo results of this study*

Term in partitioning of interaction	Size of matrix									
	4×4		6×6		8×8		12×12		16×16	
	G(1)	M(2)	G	M	G	M	G	M	G	M
1	5	6.45	9	13.35	13	20.72	21	35.39	.29	51.08
2	3	1.97	7	6.96	11	12.37	19	25.28	27	39.68
3	1	.27	5	3.17	9	7.76	17	18.85	25	31.79
4			3	1.12	7	4.39	15	13.98	23	25.78
5			1	0.15	5	2.14	13	9.96	21	20.56
6					3	0.74	11	6.84	19	16.20
7					1	.11	9	4.46	17	12.59
8							7	2.62	15	9.56
9							5	1.31	13	7.00
10							3	0.48	11	4.86
11							1	.069	9	3.26
12									7	1.93
13									5	0.97
14									3	.34
15									1	.045

(1) Gollob's formula.
(2) Monte-Carlo results (this study).

All residuals are "normalized" through division by their root mean square, to facilitate the detection of outliers.

At this time, the degrees of freedom for the breakdown of the interaction term have been calculated for matrices of size up to 20×100. Thus, the calculation of mean squares in the partitioned interaction is also limited to matrices of this size.

The author wishes to express his gratitude to Mary N. Steel for performing the Monte-Carlo calculations and for developing the program for the application of this analysis.

11. Appendix

Tables A1, A2, and A3 were obtained by a Monte-Carlo experiment, as explained in the body of the paper. Each table consists of three parts, corresponding respectively to the largest, the second-largest, and the third-largest eigenvalue. The parts are identified by the numerals 1, 2, and 3 at the beginning of each row. The second numeral in each row label represents the value of m; the column labels represent the values of n. All three tables are of course symmetrical with respect to m and n.

For interpolation purposes, use table A3 rather than A1. For example, if M_1, for $m=11$ and $n=18$ is to be calculated, we obtain by interpolation from table A3:

$$\% M_1, \text{ for } m=11,\ n=18, = 26.31.$$

The total number of degrees of freedom being $(m-1)(n-1)$, we have:

$$M_1 = 26.29\% \text{ of } (10 \times 17),$$

or

$$M_1 = 44.7.$$

TABLE A1. *Expected values of eigenvalues for random normal deviates—*M *values*

m \ *n*	4	5	6	7	8	10	12	16	20
1–4	6.45	8.47	9.86	11.61	12.88	15.08	17.96	23.33	28.20
1–5	8.47	10.37	11.82	13.59	14.75	18.11	21.01	26.29	31.87
1–6	9.86	11.82	13.35	15.44	16.92	20.42	22.85	28.87	36.78
1–7	11.61	13.59	15.44	17.18	18.91	22.89	25.59	31.83	37.46
1–8	12.88	14.75	16.92	18.91	20.72	24.15	27.67	34.52	40.70
1–10	15.08	18.11	20.42	22.89	24.15	27.81	31.88	38.81	45.11
1–12	17.96	21.01	22.85	25.59	27.67	31.88	35.39	42.76	50.00
1–16	23.33	26.29	28.87	31.83	34.52	38.81	42.76	51.08	58.11
1–20	28.20	31.87	36.78	37.46	40.70	45.11	50.00	58.11	66.34
1–32	42.86	46.77	50.83	53.77	57.50	63.21	68.60	78.62	88.71
1–50	63.85	68.93	73.74	77.62	81.68	88.66	95.11	107.41	117.36
1–100	120.90	126.58	132.94	138.45	143.64	153.00	161.58	176.32	192.08
2–4	1.97	3.04	4.01	5.06	6.06	7.81	9.98	14.08	17.90
2–5	3.04	4.25	5.45	6.72	7.78	10.10	12.50	16.80	21.27
2–6	4.01	5.45	6.96	8.21	9.68	12.15	14.60	19.51	25.54
2–7	5.06	6.72	8.21	9.77	11.21	14.26	16.77	22.18	27.02
2–8	6.06	7.78	9.68	11.21	12.73	15.70	18.72	24.26	29.77
2–10	7.81	10.10	12.15	14.26	15.70	19.21	22.12	28.60	33.86
2–12	9.98	12.50	14.60	16.77	18.72	22.12	25.28	32.43	38.32
2–16	14.08	16.80	19.51	22.18	24.26	28.60	32.43	39.68	46.05
2–20	17.90	21.27	25.54	27.02	29.77	33.86	38.32	46.05	53.76
2–32	29.99	34.01	38.38	41.15	44.63	50.22	55.54	65.32	74.28
2–50	48.10	53.54	58.35	62.61	66.57	73.16	80.12	91.31	101.51
2–100	98.31	105.69	112.21	118.49	123.48	133.58	142.62	156.95	171.26
3–4	0.27	0.67	1.15	1.72	2.26	3.39	4.87	7.74	10.66
3–5	.67	1.38	2.22	3.08	3.72	5.42	7.11	10.65	14.34
3–6	1.15	2.22	3.17	4.15	5.23	7.28	9.15	13.18	18.06
3–7	1.72	3.08	4.15	5.44	6.48	8.95	11.05	15.55	19.69
3–8	2.26	3.72	5.23	6.48	7.76	10.39	12.80	17.61	22.35
3–10	3.39	5.42	7.28	8.95	10.39	13.38	16.03	21.33	26.30
3–12	4.87	7.11	9.15	11.05	12.80	16.03	18.85	25.03	30.50
3–16	7.74	10.65	13.18	15.55	17.61	21.33	25.03	31.79	37.73
3–20	10.66	14.34	18.06	19.69	22.35	26.30	30.50	37.73	45.04
3–32	20.14	25.11	29.34	32.34	35.65	41.24	46.31	55.84	64.41
3–50	35.12	41.62	47.57	51.70	55.72	62.49	68.76	80.16	90.24
3–100	79.50	89.40	97.12	103.02	109.26	119.49	127.74	143.20	156.38

TABLE A2. *Standard deviation of eigenvalues for random normal deviates*

m \ n	4	5	6	7	8	10	12	16	20
1–4	3.24	3.83	4.05	4.36	4.52	4.72	5.26	5.71	6.25
1–5	3.83	4.18	4.12	4.28	4.71	5.12	5.45	5.84	6.40
1–6	4.05	4.12	4.30	4.88	4.71	5.64	5.51	6.09	6.92
1–7	4.36	4.28	4.88	4.80	5.16	5.28	5.64	6.03	6.51
1–8	4.52	4.71	4.71	5.16	5.26	5.30	6.02	6.49	6.88
1–10	4.72	5.12	5.64	5.28	5.30	5.50	6.41	6.25	6.89
1–12	5.26	5.45	5.51	5.64	6.02	6.41	6.65	6.57	7.53
1–16	5.71	5.84	6.09	6.03	6.49	6.25	6.57	7.10	7.60
1–20	6.25	6.40	6.92	6.51	6.88	6.89	7.53	7.60	7.87
1–32	7.38	7.59	7.96	7.89	7.98	8.02	8.26	8.37	8.77
1–50	9.55	9.55	9.44	8.88	9.04	9.36	9.77	9.86	10.16
1–100	12.96	12.22	12.50	11.84	11.96	11.67	11.59	11.95	12.19
2–4	1.39	1.77	2.08	2.25	2.62	2.77	3.17	3.56	4.11
2–5	1.77	2.04	2.29	2.48	2.70	3.11	3.43	3.95	4.32
2–6	2.08	2.29	2.61	2.83	2.94	3.36	3.68	4.00	4.46
2–7	2.25	2.48	2.83	3.04	3.04	3.36	3.77	4.20	4.72
2–8	2.62	2.70	2.94	3.04	3.17	3.43	3.82	4.26	4.75
2–10	2.77	3.11	3.36	3.36	3.43	3.99	4.09	4.66	5.00
2–12	3.17	3.43	3.68	3.77	3.82	4.09	4.10	4.85	4.93
2–16	3.56	3.95	4.00	4.20	4.26	4.66	4.85	5.30	5.55
2–20	4.11	4.32	4.46	4.72	4.75	5.00	4.93	5.55	5.83
2–32	5.46	5.24	5.60	5.44	5.52	5.82	5.88	6.27	6.20
2–50	6.93	7.09	6.79	6.96	6.61	6.92	7.08	7.36	7.20
2–100	10.15	9.59	9.08	9.56	9.04	9.23	8.90	9.27	9.41
3–4	0.37	0.69	0.92	1.20	1.35	1.72	2.13	2.80	3.14
3–5	.69	.97	1.23	1.56	1.66	1.86	2.30	2.90	3.36
3–6	.92	1.23	1.53	1.75	2.00	2.21	2.46	3.08	3.44
3–7	1.20	1.56	1.75	2.00	2.07	2.43	2.66	3.32	3.62
3–8	1.35	1.66	2.00	2.07	2.31	2.62	2.80	3.19	3.56
3–10	1.72	1.86	2.21	2.43	2.62	2.95	3.11	3.53	3.68
3–12	2.13	2.30	2.46	2.66	2.80	3.11	3.34	3.60	3.87
3–16	2.80	2.90	3.08	3.32	3.19	3.53	3.60	4.17	4.45
3–20	3.14	3.36	3.44	3.62	3.56	3.68	3.87	4.45	4.76
3–32	4.59	4.29	4.53	4.55	4.60	4.52	4.98	5.09	5.38
3–50	6.54	5.64	5.68	5.67	5.84	5.78	5.77	5.87	6.01
3–100	9.21	8.27	8.12	7.77	7.98	7.83	7.36	8.11	8.05

TABLE A3. M *Expressed as percent of total interaction degrees of freedom*

m \ n	4	5	6	7	8	10	12	16	20
1–4	74.54	69.42	65.66	62.91	60.68	57.64	54.80	51.63	49.72
1–5	69.42	63.83	59.06	55.83	53.67	50.46	47.74	44.18	41.89
1–6	65.66	59.06	54.05	51.57	48.48	44.96	42.26	38.85	36.74
1–7	62.91	55.83	51.57	47.70	45.50	41.47	38.84	35.07	32.96
1–8	60.68	53.67	48.48	45.50	42.79	38.66	36.07	32.61	30.30
1–10	57.64	50.46	44.96	41.47	38.66	34.55	32.36	28.73	26.54
1–12	54.80	47.74	42.26	38.84	36.07	32.36	29.96	25.90	24.08
1–16	51.63	44.18	38.85	35.07	32.61	28.73	25.90	22.59	20.62
1–20	49.72	41.89	36.74	32.96	30.30	26.54	24.08	20.62	18.27
1–32	46.09	38.12	32.71	29.41	26.42	22.77	20.13	16.96	14.97
1–50	43.39	35.36	29.98	26.36	23.73	20.14	17.69	14.53	12.66
1–100	40.44	32.14	26.96	23.40	20.82	17.15	14.83	11.90	10.21
2–4	22.39	25.06	26.69	27.67	28.61	29.56	30.38	31.21	31.51
2–5	25.06	26.34	27.25	27.75	28.13	28.13	28.40	28.17	27.96
2–6	26.69	27.25	28.09	27.41	27.72	26.85	26.86	26.17	25.55
2–7	27.67	27.75	27.41	27.13	26.86	26.04	25.44	24.50	23.77
2–8	28.61	28.13	27.72	26.86	26.17	25.05	24.36	22.99	22.24
2–10	29.56	28.13	26.85	26.04	25.05	23.83	22.41	21.18	19.88
2–12	30.38	28.41	26.86	25.44	24.36	22.41	21.16	19.65	18.40
2–16	31.21	28.17	26.17	24.50	22.99	21.18	19.65	17.58	16.28
2–20	31.51	27.96	25.55	23.77	22.24	19.88	18.40	16.28	14.83
2–32	32.25	27.62	24.73	22.37	20.53	18.06	16.30	14.08	12.56
2–50	32.72	27.44	23.76	21.28	19.36	16.61	14.89	12.38	10.93
2–100	32.93	26.80	22.72	20.01	17.87	14.98	13.09	10.59	9.10
3–4	3.07	5.52	7.65	9.42	10.71	12.80	14.82	17.16	18.77
3–5	5.52	8.56	11.08	12.75	13.41	15.09	16.15	17.83	18.86
3–6	7.65	11.08	12.76	13.86	14.96	16.10	16.77	17.66	18.08
3–7	9.42	12.75	13.86	15.12	15.50	16.38	16.76	17.19	17.30
3–8	10.71	13.41	14.96	15.50	15.94	16.57	16.65	16.71	16.72
3–10	12.80	15.09	16.10	16.38	16.57	16.58	16.23	15.79	15.42
3–12	14.82	16.15	16.77	16.76	16.65	16.23	15.76	15.16	14.63
3–16	17.16	17.83	17.66	17.19	16.71	15.79	15.16	14.09	13.32
3–20	18.77	18.86	18.08	17.30	16.72	15.42	14.63	13.32	12.44
3–32	21.66	20.38	18.90	17.56	16.41	14.82	13.59	12.03	10.90
3–50	23.89	21.31	19.37	17.57	16.21	14.18	12.77	10.88	9.71
3–100	26.63	22.65	19.66	17.38	15.81	13.40	11.73	9.66	8.31

12. References

[1] Church, A., Analysis of data when the response is a curve, Technometrics **8,** 229–246 (1966).

[2] Condit, H. R., and Grum, F., Spectral energy distribution of daylight, J. Opt. Soc. Am. **54,** 937–944 (1964).

[3] Gollob, H. F., A Statistical model which combines features of factor analytic and analysis of variance techniques, Psychometrika **33,** 73–116 (1968).

[4] Hotelling, H., Analysis of a complex of statistical variables into principal components, Journal of Educational Psychology **24,** 417–441; 498–520 (1933).

[5] Jeffers, J. N. R., Two case studies in the application of principal component analysis, Applied Statistics, **16,** 225–236 (1967).

[6] Judd, D. B., MacAdam, D. L., and Wyszecki, G., Spectral distribution of typical daylight as a function of correlated color temperature, J. Opt. Soc. Am. **54,** 1031–1040 (1964).

[7] Mandel, J., Non-additivity in two-way analysis of variance, J. Am. Stat. Assoc., **56,** 878–888 (1961).

[8] Mandel, J., A method for fitting empirical surfaces to physical or chemical data, Technometrics **11,** 411–429 (1969).

[9] Mandel, J. and McCrackin, F. L., Analysis of families of curves, J. Res. Nat. Bur. Stand. (U.S.), 67A, (3) 259–267 (1969).

[10] Morris, R. H., and Morrissey, J. H., An objective method for determination of equivalent neutral densities of color film images, II. Determination of primary equipment, Society of America **44,** 530–534 (1954).

[11] Pike, E. W., and Silverberg, T. R., Designing mechanical computers, Machine Design **24,** 131–137; 159–163 (1952).

[12] Simonds, J. L., Application of characteristic vector analysis to photographic and optical response data, J. Opt. Soc. Am. **53,** 968–974 (1963).

[13] Tukey, J. W., One degree of freedom for non-additivity, Biometrics **5,** 232–242 (1949).

[14] Tukey, J. W., The future of data analysis, Anns. Math. Stat. **33,** 1–67 (1962).

[15] Wernimont, G., Evaluating laboratory performance of spectrophotometers, Anal. Chem. **39,** 554-562 (1967).

[16] Wernimont, G., private communication.

[17] Williams, E. J., The interpretation of interactions in factorial experiments, Biometrika **39,** 65–81 (1952).

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Highly Restricted Partitions*

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The function $g(n, m, h, k)$, which enumerates the number of partitions of n into exactly k summands each less than or equal to m and in which the number of different summands is exactly h, is here tabulated and studied.

Key words: Generating functions; graph of a partition; partitions.

1. Introduction

The problem is to find the number $g(n, m, h, k)$ of partitions of n, into exactly k summands each $\leq m$, just h (and any h) of the natural numbers $\leq m$ being used as summands, in any partition. Evidently, we must have $h \leq k$ and also $h \leq m$.

EXAMPLE: Let $n=10$, $m=3$, $h=2$, and $k=4$. Then, there are just two acceptable partitions, viz.,

$$1, 3, 3, 3 \text{ and } 2, 2, 3, 3.$$

For $n=11$, there is only one acceptable partition: 2, 3, 3, 3. For $n \leq 4$ or $n \geq 12$, there is no partition of the desired type.

In what follows, we shall denote by $g^*(n, m, h, k)$, the number of those g-type partitions of n in which the greatest summand used is m, and by $G(n, m, h, k)$ the number of those in which the number of summands is at most k $(\geq h)$. Thus

$$g^*(n, m, h, k) = g(n, m, h, k) - g(n, m-1, h, k); \tag{1.1}$$

and

$$G(n, m, h, k) = \sum_{j=1}^{k} g(n, m, h, j). \tag{1.2}$$

2. Some Useful Relations

Let

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be the graph of a g-type partition of n. (In the graph, we have $n=17$, $h=3$, $k=6$ and we take $m=7$.) Place crosses in each row so that the total number of stars and crosses in each row is $(m+1)$. Then the graph takes the form:

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It will be readily seen that the crosses provide an acceptable partition of $k(m+1)-n$. The correspondence being one-one onto, we have:

$$g(n, m, h, k)=g(k(m+1)-n, m, h, k). \tag{2.1}$$

(The two partitions may be spoken of as complementary.)

Again, if m be the largest summand in a g-type partition of n, then in the conjugate partition (obtained by interchanging rows and columns in the graph) the largest summand is k and the number of summands exactly m. Hence

$$g^*(n, m, h, k)=g^*(n, k, h, m). \tag{2.2}$$

Similar reasoning gives

$$G(n, m, h, k)=G(n, k, h, m). \tag{2.3}$$

3. The Generating Function for g

It is not difficult to see that $g(n, m, h, k)$ is the coefficient of $x^n z^k$ in

$$\sum_{j=1}^{\binom{m}{h}} \prod_{i \in C_j} x^i z(1-x^i z)^{-1}, \tag{3.1}$$

where C_j runs through each of the $\binom{m}{h}$ sets of h distinct natural numbers $\leq m$. In other words, $g(n, m, h, k)$ is the coefficient of $x^n z^k t^h$ in

$$\prod_{i=1}^{m} \left(1+\frac{x^i z}{1-x^i z} t\right), \tag{3.2}$$

i.e., in

$$\prod_{i=1}^{m} \frac{1+x^i z(t-1)}{1-x^i z}=f(t)/f(0), \tag{3.3}$$

where

$$f(t)=1+\begin{bmatrix} m \\ 1 \end{bmatrix} z(t-1)+\begin{bmatrix} m \\ 2 \end{bmatrix} z^2(t-1)^2+\ \ldots\ +\begin{bmatrix} m \\ m \end{bmatrix} z^m(t-1)^m,$$

with

$$\begin{bmatrix} m \\ i \end{bmatrix}=\frac{(1-x^m)(1-x^{m-1})(1-x^{m-2})\ \ldots\ (1-x^{m-i+1})}{(1-x)(1-x^2)(1-x^3)\ \ldots\ (1-x^i)}\, x^{i(i+1)/2}.$$

The coefficient of t^h in $f(t)$ is

$$\begin{bmatrix} m \\ h \end{bmatrix} z^h - \binom{h+1}{1} \begin{bmatrix} m \\ h+1 \end{bmatrix} z^{h+1} + \ldots + (-1)^{m-h} \binom{m}{m-h} \begin{bmatrix} m \\ m \end{bmatrix} z^m = z^h A_h(x, z), \text{ say.} \tag{3.4}$$

Breaking into partial fractions, we have

$$A_h(x, z) \prod_{i=1}^{m} (1-x^i z)^{-1} = \sum_{i=1}^{m} B_i^{(h)}(x)/(1-x^i z), \tag{3.5}$$

with

$$B_i^{(h)}(x) = A_h(x, x^{-i})/(1-x^{-1}) \ldots (1-x^{1-i})(1-x) \ldots (1-x^{m-i}), \qquad \text{for } i \geq 2;$$

and

$$B_1^{(h)}(x) = A_h(x, x^{-i})/(1-x)(1-x^2) \ldots (1-x^{m-1}).$$

The coefficient of z^{k-h} in (3.5) is given by

$$F(x, m, h, k) = \sum_{i=1}^{m} B_i^{(h)}(x) x^{i(k-h)}. \tag{3.6}$$

Hence, $g(n, m, h, k)$ is the coefficient of x^n in $F(x, m, h, k)$.

In particular, it can be shown that

$$F(x, 3, 2, 4) = x^5 + 2x^6 + x^7 + x^8 + x^9 + 2x^{10} + x^{11}.$$

Hence,

$$g(10, 3, 2, 4) = \text{coefficient of } x^{10} \text{ in } F(x, 3, 2, 4) = 2.$$

It will be seen that the coefficient of $z^k t$ in (3.2), is

$$x^k + x^{2k} + x^{3k} + \ldots + x^{mk} = x^k(1-x^{mk})/(1-x^k).$$

Hence,

$$F(x, m, 1, k) = x^k(1-x^{mk})/(1-x^k). \tag{3.7}$$

THEOREM: *$F(x, m, h, k)$ is a polynomial in x of degree exactly $mk - \dfrac{h(h-1)}{2}$, in which the coefficients of terms equidistant from the beginning and the end are equal.*

It will be readily seen that

$$g(n, m, h, k) = 0 \text{ for } n < k + \frac{h(h-1)}{2}, \text{ and also}$$

$$\text{for } n > mk - \frac{h(h-1)}{2}.$$

Moreover,

$$g\left(mk - \frac{h(h-1)}{2}, m, h, k\right) = 1,$$

because the only g-type partition of $mk-\frac{h(h-1)}{2}$ is

$$m-1,\ m-2,\ m-3,\ldots,\ m-(h-1),\ m,\ m,\ldots,\ m.$$

The rest of the theorem, follows immediately from (2.1).

4. Reduction Formulas

We observe that $g(n, m, h, k)$ is also the number of solutions of the equation:

$$a_1x_1+a_2x_2+\ldots+a_hx_h=n, \tag{4.1}$$

such that

$$1\leq a_1<a_2<\ldots<a_h\leq m,$$

and

$$x_1+x_2+\ldots+x_h=k,$$

a's and x's being positive integers.
Since (4.1) can be written in the form:

$$(a_2-a_1)x_2+(a_3-a_1)x_3+\ldots+(a_h-a_1)x_h=n-ka_1,$$

we have for $h>1$,

$$g(n, m, h, k)=\sum_{a_1=1}^{[(n-1)/k]}\ \sum_{x_1=1}^{k-1} g(n-ka_1, m-a_1, h-1, k-x_1),$$

$$=\sum_{j=1}^{[(n-1)/k]} G(n-kj, m-j, h-1, k-1). \tag{4.2}$$

This provides a reduction formula for $g(n, m, h, k)$.
From (4.2), we readily obtain:

$$g(n, m, h, k)=G(n-k, m-1, h-1, k-1)+g(n-k, m-1, h, k), \tag{4.2'}$$

which gives

$$G(n-k, m-1, h-1, k-1)=g(n, m, h, k)-g(n-k, m-1, h, k).$$

Since

$$G(n-k, m-1, h-1, k-1)=g(n-k, m-1, h-1, k-1)+G(n-k, m-1, h-1, k-2),$$

we get

$$g(n, m, h, k)=g(n-k, m-1, h, k)+g(n-k, m-1, h-1, k-1)$$
$$+g(n-1, m, h, k-1)-g(n-k, m-1, h, k-1). \tag{4.3}$$

For $h=1$, we have

$$g(n, m, 1, k)=1, \text{ when } k|n \text{ and } n\leq km;$$
$$=0, \text{ otherwise.} \tag{4.4}$$

As an alternative proof of (4.2′), we offer the following:

The number of g-type partitions of n in which 1 appears as a summand is given by $G(n-k, m-1, h-1, k-1)$, because reducing each summand by 1, we get a partition of $(n-k)$ into at most $(k-1)$ parts with each summand $\leq (m-1)$ and using $(h-1)$ distinct natural numbers $\leq (m-1)$ as summands. The number of partitions in which 1 does not appear as a summand is given by $g(n-k, m-1, h, k)$. For, diminishing each summand by 1, as before, we are this time left with a partition of $(n-k)$ into exactly k summands each $\leq (m-1)$ and utilizing just h natural numbers $\leq (m-1)$ as summands.

Hence the result.

As an illustration of the use of (4.2), we evaluate $g(16, 6, 3, 5)$ at some length. We have

$$g(16, 6, 3, 5) = G(11, 5, 2, 4) + G(6, 4, 2, 4) + G(1, 3, 2, 4).$$

Now,

$$G(11, 5, 2, 4) = g(11, 5, 2, 4) + G(11, 5, 2, 3);$$

$$\begin{aligned} g(11, 5, 2, 4) &= G(7, 4, 1, 3) + G(3, 3, 1, 3), \\ &= 0 + g(3, 3, 1, 1) + g(3, 3, 1, 2) + g(3, 3, 1, 3), \\ &= 0 + 1 + 0 + 1 = 2. \end{aligned}$$

$$\begin{aligned} G(11, 5, 2, 3) &= g(11, 5, 2, 1) + g(11, 5, 2, 2) + g(11, 5, 2, 3), \\ &= 0 + 0 + G(8, 4, 1, 2) + G(5, 3, 1, 2) + G(2, 2, 1, 2); \end{aligned}$$

$$G(8, 4, 1, 2) = g(8, 4, 1, 1) + g(8, 4, 1, 2) = 0 + 1 = 1;$$

$$G(5, 3, 1, 2) = g(5, 3, 1, 1) + g(5, 3, 1, 2) = 0 + 0 = 0;$$

$$G(2, 2, 1, 2) = g(2, 2, 1, 1) + g(2, 2, 1, 2) = 1 + 1 = 2.$$

Again,

$$\begin{aligned} G(6, 4, 2, 4) &= g(6, 4, 2, 1) + g(6, 4, 2, 2) + g(6, 4, 2, 3) + g(6, 4, 2, 4) \\ &= 0 + G(4, 3, 1, 1) + G(2, 2, 1, 1) + G(3, 3, 1, 2) + G(2, 3, 1, 3), \end{aligned}$$

$$G(4, 3, 1, 1) = g(4, 3, 1, 1) = 0,$$

$$G(2, 2, 1, 1) = g(2, 2, 1, 1) = 1,$$

$$G(3, 3, 1, 2) = g(3, 3, 1, 1) + g(3, 3, 1, 2) = 1 + 0 = 1,$$

$$\begin{aligned} G(2, 3, 1, 3) &= g(2, 3, 1, 1) + g(2, 3, 1, 2) + g(2, 3, 1, 3) \\ &= 1 + 1 + 0 = 2. \end{aligned}$$

Finally,

$$G(1, 3, 2, 4) = 0.$$

Hence

$$g(16, 6, 3, 5) = 9.$$

Tracing backward, each one of the nine paths of reduction, this method enables us to write down the corresponding partitions as well.

Thus, one path of reduction is:

$$g(2, 2, 1, 1) \leftarrow g(11, 5, 2, 3) \leftarrow g(16, 6, 3, 5).$$

This gives in (4.1),

$$a_1 = 1,\ x_1 = 2;\ a_2 - a_1 = 3,\ x_2 = 2;\ a_3 - a_2 = 2,\ x_3 = 1;$$

which leads to the partition: 1, 1, 4, 4, 6 of 16. The other partitions viz.,

$$1, 1, 2, 6, 6;\ 1, 1, 4, 5, 5;\ 1, 3, 3, 3, 6;\ 1, 3, 4, 4, 4;$$
$$2, 3, 3, 4, 4;\ 2, 3, 3, 3, 5;\ 2, 2, 3, 3, 6;\ 2, 2, 2, 4, 6;$$

can be similarly obtained.

As another reduction formula, we have the following:

$$g(n, m, h, k) = g(n, m-1, h, k) + \sum_{j=1}^{[(n-1)/m]} g(n-jm, m-1, h-1, k-j), \tag{4.5}$$

because $g(n, m-1, h, k)$ is the number of those partitions of n in which m does not at all occur as a part, while the sigma provides the rest.

From (4.5), we can easily deduce that

$$g(n, m, h, k) = g(n, m-1, h, k) + g(n-m, m-1, h-1, k-1) + g(n-m, m, h, k-1) - g(n-m, m-1, h, k-1). \tag{4.6}$$

When $m \neq k$, (2.2) gives the simpler result:

$$\begin{aligned} g(n, m, h, k) &= g(n, m-1, h, k) + g^*(n, m, h, k), \\ &= g(n, m-1, h, k) + g^*(n, k, h, m), \\ &= g(n, m-1, h, k) + g(n, k, h, m) - g(n, k-1, h, m). \end{aligned} \tag{4.7}$$

In computing the tables that are appended, we have made use of (4.6). The other formulas have served as checks. In view of (2.1), the tables give values of $g(n, m, h, k)$ for

$$k + \frac{h(h-1)}{2} \leq n \leq [k(m+1)/2]; \qquad m, h, k \leq 10.$$

The plan of the tables is easy to follow. For even values of h, the values of $g(n, m, h, k)$ appear in the right-hand triangles and those for odd values of h in the left-hand ones.

For fixed n, h, k, the values of $g(n, m, h, k)$ are nondecreasing for increasing m. This is what could be expected. But, for fixed m, h, k and increasing n, the function $g(n, m, h, k)$ behaves quite irregularly. This is rather unusual.

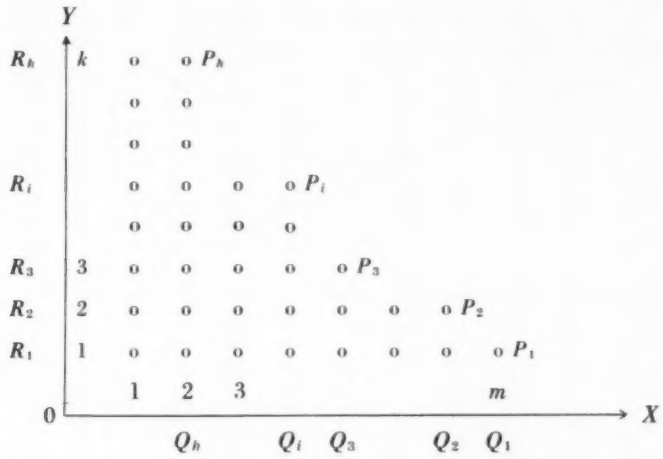
A close study of the tables, has led to the

THEOREM: *For any fixed m, h and k,*

$$\sum_{n=1}^{\infty} g^*(n, m, h, k) = \binom{m-1}{h-1}\binom{k-1}{h-1}. \tag{4.8}$$

We offer a combinatorial proof.

Let



be the graph of a g^*-partition for some n. The graph determines uniquely and is uniquely determined by its corner points: $P_1, P_2, P_3, \ldots, P_h$. These, in turn, are uniquely determined by and uniquely determine, two sets of h points each viz., $Q=\{Q_1, Q_2, \ldots, Q_h\}$, and $R=\{R_1, R_2, \ldots, R_h\}$ obtained from the projection of the points $P_1, P_2, \ldots, P_h$ on the axes of X and Y. The points $Q_2, Q_3, \ldots, Q_h$ lie strictly between 0 and m $(=Q_1)$ on the X axis. The points $R_1, R_2, \ldots, R_{h-1}$ lie strictly between 0 and k $(=R_h)$ on the Y axis. The total number of ways in which the $(h-1)$ points $Q_2, Q_3, \ldots, Q_h$ can be placed between 0 and m, is $\binom{m-1}{h-1}$. Similarly, the $(h-1)$ points $R_1, R_2, \ldots, R_{h-1}$ can be located between 0 and k on the Y axis in $\binom{k-1}{h-1}$ ways. Hence the total number of g^*-partitions must be $\binom{m-1}{h-1}\binom{k-1}{h-1}$ and the theorem follows.

Proceeding on the same lines, one can prove that

$$\sum_{n=1}^{\infty} g(n, m, h, k) = \binom{k-1}{h-1}\binom{m}{h}, \tag{4.9}$$

and

$$\sum_{n=1}^{\infty} G(n, m, h, k) = \binom{k}{h}\binom{m}{h}. \tag{4.10}$$

Result (4.9) can be derived inductively from (4.6). The results (4.9) and (4.10) can both be derived also from the theorem itself.

The points of sets Q and R determine uniquely another set of h points $T=\{T_1, T_2, \ldots, T_h\}$; where T_j is the point of which Q_{h-j+1} is the projection on the X axis and R_j the projection on the Y axis.

The broken line $OT_1\ T_2 \ldots T_h$ corresponds to an ordered partition of the vector (m, k) into exactly h nondegenerate parts.[1] Hence the sum in (4.8) can be interpreted as the number of such partitions of the bipartite number (m, k).

[1] Gupta, H., Graphic Representation of a Partition of a j-Partite Number. Research Bulletin (N.S.), Panjab University, **10**, 189–196 (1959).

$h=2$

***	**	**	**	**	**	**	**	**	**	**	***	
		2	3	4	5	6	7	8	9	10	m / n	k
		1	1	1	1	1	1	1	1	1	3	2
			1	1	1	1	1	1	1	1	4	
				2	2	2	2	2	2	2	5	
					2	2	2	2	2	2	6	
						3	3	3	3	3	7	
							3	3	3	3	8	
								4	4	4	9	
									4	4	10	
										5	11	
		1	1	1	1	1	1	1	1	1	8	7
		1	2	2	2	2	2	2	2	2	9	
		1	1	2	2	2	2	2	2	2	10	
			2	2	3	3	3	3	3	3	11	
38	47		1	1	1	2	2	2	2	2	12	
37	54		2	3	3	3	4	4	4	4	13	
36	51		0	0	0	0	0	1	1	1	14	
35	54	50		2	3	3	3	3	4	4	15	
34	49	38		3	3	3	3	3	3	4	16	
33	47	40		2	3	4	4	4	4	4	17	
32	45	35			2	3	3	3	3	3	18	
31	47	38	28		4	4	6	6	6	6	19	
30	44	38	30		3	3	3	4	4	4	20	
29	45	40	31		0	0	0	1	2	2	21	
28	47	39	37	31		5	5	5	5	6	22	
27	39	35	26	21		4	4	4	5	5	23	
26	34	28	24	18		3	4	4	4	4	24	
25	38	35	28	23			5	5	5	6	25	
24	34	29	25	18	16		4	6	6	6	26	
23	33	31	27	22	14		5	5	6	6	27	
22	26	23	21	16	11		0	1	2	3	28	
21	32	30	27	26	20	14		5	5	5	29	
20	24	21	19	16	14	8		4	4	5	30	
19	21	19	16	14	12	9		6	7	7	31	
18	20	18	16	13	11	8			6	6	32	
17	18	17	15	13	10	9	5		6	6	33	
16	14	14	13	11	9	6	4		5	7	34	
15	12	12	12	11	9	7	5		2	3	35	
14	11	11	11	11	10	8	6	3		5	36	
13	6	6	6	6	6	5	3	2		6	37	
12	5	5	5	5	5	5	4	2		7	38	
11	2	2	2	2	2	2	2	1				
10	1	1	1	1	1	1	1	1				
***												
n / m	10	9	8	7	6	5	4	3				

$h=3$

$h=2$

***	**	**	**	**	**	**	**	**	**	**	***	
		2	3	4	5	6	7	8	9	10	m / n	k
		1	1	1	1	1	1	1	1	1	4	3
			2	2	2	2	2	2	2	2	5	
16	10		0	1	1	1	1	1	1	1	6	
15	10	8		2	3	3	3	3	3	3	7	
14	9	8			2	3	3	3	3	3	8	
13	8	7	6		2	2	3	3	3	3	9	
12	7	7	6	5		3	3	4	4	4	10	
11	5	5	5	4			4	4	5	5	11	
10	4	4	4	4	3		2	3	3	4	12	
9	3	3	3	3	3	2		4	5	5	13	
8	2	2	2	2	2	2			4	5	14	
7	1	1	1	1	1	1	1		4	4	15	
6	1	1	1	1	1	1	1	1		5	16	
		1	1	1	1	1	1	1	1	1	7	6
		1	2	2	2	2	2	2	2	2	8	
		1	1	2	2	2	2	2	2	2	9	
33	32		2	2	3	3	3	3	3	3	10	
32	44		1	1	1	2	2	2	2	2	11	
31	42		1	2	2	2	3	3	3	3	12	
30	38	30		1	1	1	1	2	2	2	13	
29	40	33		3	4	4	4	4	5	5	14	
28	41	35		2	3	3	3	3	3	4	15	
27	32	24	18		3	5	5	5	5	5	16	
26	38	32	25		1	1	2	2	2	2	17	
25	36	33	27		4	4	5	6	6	6	18	
24	31	27	23	18		1	1	1	2	2	19	
23	34	29	24	20		4	4	5	5	6	20	
22	29	26	22	16		5	5	5	5	5	21	
21	24	20	17	12	6		6	6	7	7	22	
20	26	25	21	19	13		1	2	2	2	23	
19	26	23	22	18	15		5	6	7	8	24	
18	19	17	14	13	10	6		1	1	2	25	
17	21	19	17	14	13	9		6	7	7	26	
16	16	15	13	11	8	8		6	6	6	27	
15	13	13	12	10	8	5	4		6	7	28	
14	10	10	10	9	7	5	3		2	2	29	
13	10	10	10	10	9	7	5		8	9	30	
12	6	6	6	6	6	5	3	2		2	31	
11	5	5	5	5	5	5	4	2		7	32	
10	2	2	2	2	2	2	2	1		7	33	
9	1	1	1	1	1	1	1	1				
n / m	10	9	8	7	6	5	4	3				

$h=3$

$h=2$

***	**	**	**	**	**	**	**	**	**	**	*** m \ n	k
		2	3	4	5	6	7	8	9	10		
		1	1	1	1	1	1	1	1	1	5	4
		1	2	2	2	2	2	2	2	2	6	
22	16		1	2	2	2	2	2	2	2	7	
21	22		1	1	2	2	2	2	2	2	8	
20	18	14		1	1	2	2	2	2	2	9	
19	20	17		4	4	4	5	5	5	5	10	
18	16	13	10		2	2	2	3	3	3	11	
17	19	16	13		2	3	3	3	4	4	12	
16	14	13	10	7		2	3	3	3	4	13	
15	16	14	13	10		5	5	6	6	6	14	
14	11	10	8	7	4		2	2	3	3	15	
13	11	11	10	8	7		5	5	5	6	16	
12	8	8	8	7	5	4		3	3	3	17	
11	7	7	7	7	6	4		6	7	7	18	
10	3	3	3	3	3	2	0		3	4	19	
9	4	4	4	4	4	4	3		6	6	20	
8	2	2	2	2	2	2	2	1		3	21	
7	1	1	1	1	1	1	1	1		9	22	
		1	1	1	1	1	1	1	1	1	6	5
		1	2	2	2	2	2	2	2	2	7	
			1	2	2	2	2	2	2	2	8	
27	31		2	2	3	3	3	3	3	3	9	
26	28		0	0	0	1	1	1	1	1	10	
25	31	26		3	3	3	4	4	4	4	11	
24	29	22		2	2	2	2	3	3	3	12	
23	28	25			4	4	4	4	5	5	13	
22	25	21	17		3	4	4	4	4	5	14	
21	25	22	19		0	1	2	2	2	2	15	
20	26	21	18	14		3	3	4	4	4	16	
19	23	21	16	14		4	5	5	6	6	17	
18	20	18	16	11			4	4	4	5	18	
17	18	16	14	12	8		4	5	5	5	19	
16	17	15	13	11	9		2	3	3	3	20	
15	18	17	15	13	11	10		4	6	6	21	
14	11	11	10	8	6	4		5	5	6	22	
13	9	9	9	8	6	4			5	6	23	
12	8	8	8	8	7	5	3		5	5	24	
11	5	5	5	5	5	4	2		4	4	25	
10	5	5	5	5	5	5	4	2		7	26	
9	2	2	2	2	2	2	2	1		5	27	
8	1	1	1	1	1	1	1	1				
n \ m	10	9	8	7	6	5	4	3				

$h=3$

$h=2$

***	**	** 2	** 3	** 4	** 5	** 6	** 7	** 8	** 9	** 10	*** m / n	k
		1	1	1	1	1	1	1	1	1	9	8
		1	2	2	2	2	2	2	2	2	10	
		1	1	2	2	2	2	2	2	2	11	
		1	2	2	3	3	3	3	3	3	12	
44	56		1	1	1	2	2	2	2	2	13	
43	54		2	3	3	3	4	4	4	4	14	
42	72		1	1	1	1	1	2	2	2	15	
41	61		1	1	2	2	2	2	3	3	16	
40	66	56		2	2	2	2	2	2	3	17	
39	55	45		3	3	4	4	4	4	4	18	
38	66	56		1	2	2	2	2	2	2	19	
37	51	41		4	5	6	7	7	7	7	20	
36	52	40	26		1	1	2	2	2	2	21	
35	56	48	38		4	4	4	6	6	6	22	
34	63	54	42		2	3	3	3	4	4	23	
33	48	41	33		2	3	3	3	4	5	24	
32	56	48	40	29		2	2	2	2	2	25	
31	43	39	32	23		4	6	6	6	7	26	
30	53	46	42	32		2	2	2	2	2	27	
29	43	35	29	28		7	8	9	9	9	28	
28	40	35	26	20	14		2	4	4	4	29	
27	41	36	31	23	18		4	4	6	6	30	
26	41	38	32	26	20		3	3	3	4	31	
25	37	33	29	23	18		5	5	6	7	32	
24	35	34	31	26	19	14		3	3	3	33	
23	29	25	24	20	15	11		6	6	6	34	
22	28	26	22	22	17	11		2	2	3	35	
21	27	24	22	18	17	12		10	11	11	36	
20	20	18	15	13	10	8	4		4	4	37	
19	22	20	18	15	13	9	7		6	8	38	
18	19	18	16	14	11	10	5		3	3	39	
17	16	16	15	13	11	8	6		6	7	40	
16	13	13	13	12	10	8	6	3		3	41	
15	11	11	11	11	10	8	6	3		6	42	
14	6	6	6	6	6	5	3	2		4	43	
13	5	5	5	5	5	5	4	2		13	44	
12	2	2	2	2	2	2	2	1				
11	1	1	1	1	1	1	1	1				
n / m	10	9	8	7	6	5	4	3				

$h=3$

$h=2$

***	**	**	**	**	**	**	**	**	**	**	***	
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		1	1	1	1	1	1	1	1	1	10	9
		1	2	2	2	2	2	2	2	2	11	
		1	1	2	2	2	2	2	2	2	12	
		1	2	2	3	3	3	3	3	3	13	
			1	1	1	2	2	2	2	2	14	
49	73		2	3	3	3	4	4	4	4	15	
48	65		1	1	1	1	1	2	2	2	16	
47	74		2	2	3	3	3	3	4	4	17	
46	74		0	1	1	1	1	1	1	2	18	
45	82	72		2	2	3	3	3	3	3	19	
44	66	51		2	2	2	2	2	2	2	20	
43	70	60		3	5	5	6	6	6	6	21	
42	63	49		2	2	3	3	3	3	3	22	
41	67	60			2	2	3	4	4	4	23	
40	70	57	47		4	5	5	6	6	6	24	
39	58	46	38		3	3	3	3	5	5	25	
38	66	52	42		2	3	3	3	3	4	26	
37	64	57	39		2	2	3	3	3	4	27	
36	64	55	48	34		2	3	3	3	3	28	
35	64	58	46	36		4	4	4	4	4	29	
34	51	46	40	28		6	6	8	8	8	30	
33	48	40	34	26		2	3	3	3	3	31	
32	51	43	39	31			2	3	4	4	32	
31	56	51	41	34	24		8	8	10	10	33	
30	41	32	28	24	16		4	4	4	6	34	
29	50	46	36	30	21		3	3	3	3	35	
28	43	38	34	25	23		2	3	3	4	36	
27	44	42	36	31	22	16		5	5	5	37	
26	36	33	30	24	20	13		4	4	4	38	
25	36	34	32	28	21	15		8	10	10	39	
24	26	22	20	17	13	9		3	3	3	40	
23	31	29	25	24	20	15			5	6	41	
22	26	23	21	17	15	11	7		8	10	42	
21	22	20	17	15	12	9	6		4	4	43	
20	23	21	19	16	14	10	7		5	5	44	
19	21	20	18	16	13	12	7		4	5	45	
18	17	17	16	14	12	9	7	4		5	46	
17	13	13	13	12	10	8	6	3		4	47	
16	11	11	11	11	10	8	6	3		10	48	
15	6	6	6	6	6	5	3	2		5	49	
14	5	5	5	5	5	5	4	2				
13	2	2	2	2	2	2	2	1				
12	1	1	1	1	1	1	1	1				
n / m	10	9	8	7	6	5	4	3				

$h=3$

$h=2$

***	**	**	**	**	**	**	**	**	**	**	***	
		2	3	4	5	6	7	8	9	10	m / n	k
		1	1	1	1	1	1	1	1	1	11	10
		1	2	2	2	2	2	2	2	2	12	
		1	1	2	2	2	2	2	2	2	13	
		1	2	2	3	3	3	3	3	3	14	
		1	1	1	1	2	2	2	2	2	15	
55	56		2	3	3	3	4	4	4	4	16	
54	97		1	1	1	1	1	2	2	2	17	
53	78		2	2	3	3	3	3	4	4	18	
52	91		1	2	2	2	2	2	2	3	19	
51	74		1	1	1	2	2	2	2	2	20	
50	84	68		1	1	1	1	1	1	1	21	
49	84	62		4	5	5	6	6	6	6	22	
48	87	69		1	2	2	2	2	2	2	23	
47	76	64		3	3	4	4	5	5	5	24	
46	89	77		2	2	3	4	4	4	4	25	
45	51	42	32		5	5	5	6	7	7	26	
44	78	69	54		1	1	1	1	2	2	27	
43	78	60	50		4	5	6	6	6	8	28	
42	79	67	53		2	2	2	2	2	2	29	
41	64	56	49		2	3	4	4	4	4	30	
40	82	65	52	42		2	2	3	3	3	31	
39	63	55	43	33		5	5	6	6	6	32	
38	69	61	47	37		2	2	2	2	2	33	
37	59	54	47	37		5	7	7	9	9	34	
36	65	59	50	38		3	4	5	5	5	35	
35	47	40	34	26	22		5	5	6	7	36	
34	60	50	45	36	25		2	2	2	4	37	
33	51	45	37	30	23		6	8	8	8	38	
32	57	51	43	35	25		2	2	2	2	39	
31	50	42	37	33	24		5	6	6	6	40	
30	52	48	39	32	26	18		2	3	3	41	
29	43	39	35	27	22	15		6	8	8	42	
28	41	40	35	30	22	16		2	2	2	43	
27	38	34	32	27	23	15		7	7	9	44	
26	33	31	28	25	19	13		6	7	7	45	
25	29	25	23	19	16	13	8		7	9	46	
24	30	28	24	23	18	14	7		2	2	47	
23	28	25	23	19	17	12	9		8	8	48	
22	23	21	18	16	13	10	6		2	2	49	
21	25	23	21	18	16	12	9		8	8	50	
20	22	21	19	17	14	13	8	4		3	51	
19	17	17	16	14	12	9	7	4		10	52	
18	13	13	13	12	10	8	6	3		2	53	
17	11	11	11	11	10	8	6	3		7	54	
16	6	6	6	6	6	5	3	2		9	55	
15	5	5	5	5	5	5	4	2				
14	2	2	2	2	2	2	2	1				
13	1	1	1	1	1	1	1	1				
n / m	10	9	8	7	6	5	4	3				

$h=3$

$h=4$

***	***	*** 4	*** 5	*** 6	*** 7	*** 8	*** 9	*** 10	*** m / n	k
		1	1	1	1	1	1	1	10	4
			1	1	1	1	1	1	11	
			1	2	2	2	2	2	12	
				2	3	3	3	3	13	
				3	4	5	5	5	14	
					4	5	6	6	15	
					5	7	8	9	16	
						7	9	10	17	
						8	11	13	18	
							11	14	19	
							12	16	20	
								16	21	
								18	22	
		1	1	1	1	1	1	1	13	7
		1	2	2	2	2	2	2	14	
		2	4	5	5	5	5	5	15	
		3	7	9	10	10	10	10	16	
		3	6	11	13	14	14	14	17	
			9	14	19	21	22	22	18	
38	170		8	15	21	26	28	29	19	
37	176		10	16	25	31	36	38	20	
36	168		6	13	20	30	36	41	21	
35	164	104		23	32	41	51	57	22	
34	159	102		21	31	41	51	61	23	
33	143	92		20	38	50	62	72	24	
32	136	94			35	51	64	77	25	
31	129	85	50		43	61	79	94	26	
30	116	87	50		42	65	81	100	27	
29	103	76	51		26	53	80	98	28	
28	98	75	53	35		62	88	113	29	
27	76	59	38	20		67	92	122	30	
26	64	52	36	20		69	100	125	31	
25	56	47	35	21			103	137	32	
24	43	38	29	18	9		105	143	33	
23	33	31	26	17	8		103	141	34	
22	22	21	19	14	6		94	144	35	
21	17	17	16	14	9	3		150	36	
20	9	9	9	8	6	2		146	37	
19	5	5	5	5	4	2		158	38	
18	2	2	2	2	2	1				
17	1	1	1	1	1	1				
n / m	10	9	8	7	6	5				

$h=5$

$h=4$

***	***	*** 4	*** 5	*** 6	*** 7	*** 8	**** 9	*** 10	*** m \ n	k
		1	1	1	1	1	1	1	11	5
		1	2	2	2	2	2	2	12	
			3	4	4	4	4	4	13	
27	20		4	6	7	7	7	7	14	
26	19		0	4	6	7	7	7	15	
25	18	12		6	10	12	13	13	16	
24	16	11		7	10	14	16	17	17	
23	14	11			13	16	20	22	18	
22	11	9	6		11	18	21	25	19	
21	9	8	6		12	19	26	29	20	
20	7	6	5	3		19	27	34	21	
19	5	5	4	3		21	31	39	22	
18	3	3	3	2			29	40	23	
17	2	2	2	2	1		33	43	24	
16	1	1	1	1	1		30	44	25	
15	1	1	1	1	1	1		47	26	
								46	27	
		1	1	1	1	1	1	1	12	6
		1	2	2	2	2	2	2	13	
		2	4	5	5	5	5	5	14	
		2	6	8	9	9	9	9	15	
33	66		4	9	11	12	12	12	16	
32	75		5	9	14	16	17	17	17	
31	68		6	11	16	21	23	24	18	
30	69	42		10	17	22	27	29	19	
29	61	40		10	16	24	29	34	20	
28	62	42		20	27	35	43	48	21	
27	50	34	18		23	31	40	48	22	
26	50	37	23		23	37	47	56	23	
25	39	32	20		22	35	50	61	24	
24	35	28	21	11		39	53	70	25	
23	27	23	16	10		37	53	68	26	
22	22	20	16	9		48	68	85	27	
21	13	12	10	6	0		56	76	28	
20	12	12	11	9	5		64	89	29	
19	7	7	7	6	4		62	86	30	
18	4	4	4	4	3	1		93	31	
17	2	2	2	2	2	1		85	32	
16	1	1	1	1	1	1		104	33	
n \ m	10	9	8	7	6	5				

$h=5$

$h=4$

***	***	***	***	***	***	***	***	***	***	k
		4	5	6	7	8	9	10	m / n	
		1	1	1	1	1	1	1	14	8
		1	2	2	2	2	2	2	15	
		2	4	5	5	5	5	5	16	
		3	7	9	10	10	10	10	17	
		4	7	12	14	15	15	15	18	
		4	11	16	21	23	24	24	19	
		5	12	20	26	31	33	34	20	
44	324		12	20	30	36	41	43	21	
43	337		12	20	29	40	46	51	22	
42	339		14	28	38	49	60	66	23	
41	340		11	24	36	47	59	70	24	
40	321	200		30	46	60	73	85	25	
39	311	196		26	44	59	74	88	26	
38	297	194		33	56	75	92	109	27	
37	279	181		33	60	82	100	118	28	
36	255	164	78		53	89	113	133	29	
35	256	183	106		50	75	108	131	30	
34	237	170	107		64	96	127	162	31	
33	204	151	92		55	85	122	151	32	
32	194	142	96	47		103	140	181	33	
31	162	126	83	43		86	127	166	34	
30	152	116	85	48		97	139	187	35	
29	127	99	68	46		118	160	203	36	
28	108	86	60	34	16		161	215	37	
27	89	75	54	33	15		143	200	38	
26	77	67	53	34	18		158	223	39	
25	56	51	41	28	14		144	212	40	
24	43	41	36	26	15	5		221	41	
23	27	26	24	19	10	4		212	42	
22	19	19	18	16	11	4		239	43	
21	10	10	10	9	7	3		236	44	
20	5	5	5	5	4	2				
19	2	2	2	2	2	1				
18	1	1	1	1	1	1				
n / m	10	9	8	7	6	5				

$h=5$

$h=4$

***	***	***	***	***	***	***	***	***	***	
		4	5	6	7	8	9	10	m / n	k
		1	1	1	1	1	1	1	15	9
		1	2	2	2	2	2	2	16	
		2	4	5	5	5	5	5	17	
		3	7	9	10	10	10	10	18	
		4	7	12	14	15	15	15	19	
		5	12	17	22	24	25	25	20	
		6	14	22	28	33	35	36	21	
		6	16	25	35	41	46	48	22	
			14	24	34	45	51	56	23	
49	581		20	35	47	59	70	76	24	
48	580		16	30	43	56	69	80	25	
47	573		19	35	53	68	83	96	26	
46	577		16	36	52	69	85	101	27	
45	590	378		38	65	83	102	120	28	
44	520	326		41	67	91	108	128	29	
43	507	314		47	72	105	132	151	30	
42	508	334		41	72	103	131	157	31	
41	471	308			75	112	147	178	32	
40	460	316	177		85	122	165	196	33	
39	419	278	172		82	123	165	210	34	
38	407	285	165		79	121	168	213	35	
37	366	260	152		74	116	164	216	36	
36	354	270	174	88		143	187	240	37	
35	305	227	154	77		139	192	245	38	
34	272	211	141	78		140	215	270	39	
33	240	179	127	68		134	197	266	40	
32	216	170	119	73			211	285	41	
31	189	148	107	64	29		210	287	42	
30	163	130	94	63	26		222	304	43	
29	138	114	83	52	27		223	316	44	
28	112	97	74	48	26		184	280	45	
27	91	81	66	45	24	8		306	46	
26	66	61	51	37	21	7		317	47	
25	48	46	41	31	19	7		323	48	
24	29	28	26	21	12	5		321	49	
23	20	20	19	17	12	5				
22	10	10	10	9	7	3				
21	5	5	5	5	4	2				
20	2	2	2	2	2	1				
19	1	1	1	1	1	1				
n / m	10	9	8	7	6	5				

$h=5$

362-120 O - 69 - 5

$h=4$

***	***	*** 4	*** 5	*** 6	*** 7	*** 8	*** 9	*** 10	*** m / n	k
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		1	2	2	2	2	2	2	17	
		2	4	5	5	5	5	5	18	
		3	7	9	10	10	10	10	19	
		4	7	12	14	15	15	15	20	
		5	12	17	22	24	25	25	21	
		7	15	23	29	34	36	37	22	
		7	18	27	37	43	48	50	23	
		8	18	29	39	50	56	61	24	
		8	22	39	52	64	75	81	25	
55	864		22	37	52	66	79	90	26	
54	940		24	41	60	77	93	106	27	
53	955		22	42	60	78	96	113	28	
52	895		25	50	75	95	115	135	29	
51	915		22	46	76	99	118	139	30	
50	855	514		54	79	114	140	161	31	
49	897	541		51	85	114	145	170	32	
48	825	511		60	98	139	170	204	33	
47	833	540		54	92	131	176	204	34	
46	771	523		62	112	160	205	250	35	
45	714	468	256		101	144	196	242	36	
44	692	466	272		101	155	208	266	37	
43	700	474	281		106	163	216	272	38	
42	632	439	265		112	177	232	298	39	
41	603	431	268		100	168	237	288	40	
40	558	393	249	128		175	249	326	41	
39	512	380	239	124		175	252	326	42	
38	462	345	223	120		179	261	332	43	
37	424	329	223	125		177	270	361	44	
36	375	287	203	112		212	308	411	45	
35	324	253	175	102	42		260	367	46	
34	302	231	170	103	46		274	384	47	
33	257	206	145	92	41		288	393	48	
32	230	184	138	86	44		285	386	49	
31	195	160	119	83	39		292	418	50	
30	162	137	104	68	39	12		419	51	
29	126	111	87	59	32	11		414	52	
28	101	91	76	54	31	11		416	53	
27	71	66	56	42	25	9		404	54	
26	50	48	43	33	21	8		476	55	
25	30	29	27	22	13	6				
24	20	20	19	17	12	5				
23	10	10	10	9	7	3				
22	5	5	5	5	4	2				
21	2	2	2	2	2	1				
20	1	1	1	1	1	1				
n / m	10	9	8	7	6	5				

$h=5$

$h = 6$

***	***	*** 6	*** 7	*** 8	*** 9	*** 10	*** m / n	k
		1	1	1	1	1	21	6
			1	1	1	1	22	
			1	2	2	2	23	
			1	2	3	3	24	
				3	4	5	25	
				3	5	6	26	
				4	7	9	27	
					7	10	28	
					8	13	29	
					8	14	30	
						16	31	
						16	32	
						18	33	
		1	1	1	1	1	24	9
		1	2	2	2	2	25	
		2	4	5	5	5	26	
		3	7	9	10	10	27	
		4	11	16	18	19	28	
		5	16	25	30	32	29	
		6	16	32	42	47	30	
		6	22	41	59	69	31	
			22	48	72	91	32	
49	171		28	57	93	119	33	
48	170		26	62	103	144	34	
47	167		29	67	121	172	35	
46	165		24	66	122	189	36	
45	164	84		89	154	230	37	
44	131	56		88	165	253	38	
43	123	56		88	189	293	39	
42	116	57		88	190	315	40	
41	101	52			210	353	41	
40	90	49	18		209	372	42	
39	74	42	17		231	413	43	
38	63	38	15		233	441	44	
37	48	32	12		186	416	45	
36	39	30	16	4		450	46	
35	25	20	12	3		473	47	
34	16	14	9	3		482	48	
33	9	8	6	2		489	49	
32	5	5	4	2				
31	2	2	2	1				
30	1	1	1	1				
n / m	10	9	8	7				

$h = 7$

$h=6$

***	***	***	***	***	***	***	***	
		6	7	8	9	10	m / n	k
		1	1	1	1	1	22	7
		1	2	2	2	2	23	
		1	3	4	4	4	24	
			4	6	7	7	25	
38	10		5	9	11	12	26	
37	10		6	12	16	18	27	
36	9		0	9	16	20	28	
35	8	4		12	23	30	29	
34	7	4		14	24	36	30	
33	5	3		15	31	43	31	
32	4	3			31	51	32	
31	3	2	1		36	59	33	
30	2	2	1		33	61	34	
29	1	1	1		34	67	35	
28	1	1	1	1		71	36	
						72	37	
						76	38	
		1	1	1	1	1	23	8
		1	2	2	2	2	24	
		2	4	5	5	5	25	
		2	6	8	9	9	26	
		3	9	14	16	17	27	
		3	12	20	25	27	28	
44	48		9	23	32	37	29	
43	54		11	25	41	50	30	
42	50		13	30	49	66	31	
41	52		13	32	58	79	32	
40	44	20		36	65	96	33	
39	43	20		35	71	109	34	
38	34	18		35	73	120	35	
37	32	16		56	98	151	36	
36	23	12	0		95	156	37	
35	23	16	7		97	176	38	
34	16	12	6		96	179	39	
33	11	9	5		98	197	40	
32	7	6	4	1		193	41	
31	4	4	3	1		211	42	
30	2	2	2	1		210	43	
29	1	1	1	1		228	44	
n / m	10	9	8	7				

$h=7$

$h=6$

***	***	***	***	***	***	***	***	
		6	7	8	9	10	m / n	k
		1	1	1	1	1	25	10
		1	2	2	2	2	26	
		2	4	5	5	5	27	
		3	7	9	10	10	28	
		5	12	17	19	20	29	
		6	18	27	32	34	30	
		8	20	37	47	52	31	
		9	29	50	69	79	32	
		11	34	65	91	111	33	
		11	39	77	118	146	34	
		12	45	92	143	189	35	
55	388		49	102	169	230	36	
54	458		47	108	184	265	37	
53	454		54	133	221	319	38	
52	420		53	136	240	356	39	
51	418		54	153	278	413	40	
50	385	154		150	293	451	41	
49	380	159		168	331	520	42	
48	342	152		162	344	549	43	
47	326	156		179	387	639	44	
46	281	150		182	412	690	45	
45	248	122	42		396	733	46	
44	221	116	41		404	740	47	
43	206	112	45		441	814	48	
42	170	98	39		429	811	49	
41	148	89	40		452	893	50	
40	115	73	32	8		880	51	
39	93	63	31	8		938	52	
38	66	48	23	7		900	53	
37	49	39	23	7		930	54	
36	30	25	16	5		1020	55	
35	18	16	11	4				
34	10	9	7	3				
33	5	5	4	2				
32	2	2	2	1				
31	1	1	1	1				
n / m	10	9	8	7				

$h=7$

$h=8$

***	**	**	**	**	***	
		8	9	10	m / n	k
		1	1	1	36	8
			1	1	37	
			1	2	38	
			1	2	39	
			1	3	40	
				3	41	
				4	42	
				4	43	
				5	44	
		1	1	1	37	9
		1	2	2	38	
		1	3	4	39	
		1	4	6	40	
			5	9	41	
49	1		6	12	42	
48	1		7	16	43	
47	1		8	20	44	
46	1		0	16	45	
45	1	1		20	46	
				23	47	
				25	48	
				26	49	
		1	1	1	38	10
		1	2	2	39	
		2	4	5	40	
		2	6	8	41	
		3	9	14	42	
		3	12	20	43	
		4	16	30	44	
		4	20	40	45	
55	0		16	46	46	
54	9		19	51	47	
53	8		22	60	48	
52	7		23	66	49	
51	6		24	75	50	
50	5	1		79	51	
49	4	1		85	52	
48	3	1		84	53	
47	2	1		84	54	
46	1	1		120	55	
n / m	10	9				

$h=9$

$h=10 \mid =k$

10	m / n
1	55

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Publications of the National Bureau of Standards*

Selected Abstracts

Haber, S., Osgood, C. F., **On a theorem of Piatetsky-Shapiro and approximation of multiple integrals,** *Math. Comput.* **23,** *No. 105, 165–168 (Jan. 1969).*

Key words: Banach-Steinhaus theorem; fourier series; multiple integrals; numerical analysis; number theory; periodic functions; quadrature.

Quadrature formulas are derived for certain classes of functions of several variables having absolutely convergent Fourier Series. A limitation on formulas of the type considered is proven.

Newman, M., **Maximal normal subgroups of the modular group,** *Proc. Am. Math. Soc.* **19,** *No. 5, 1138–1144 (Oct. 1968).*

Key words: Linear fractional groups; maximal normal subgroups; simple groups.

The principal results of this paper are: (1) The modular group Γ contains infinitely many maximal normal subgroups of finite index which are not congruence groups, and (2) any element of $LF(2, q)$ other than $\pm I$ may be written as the product of an element of period 2 and an element of period 3, provided that $p > 3$, where $q = p^n$.

Olver, F. W. J., **Error bounds for the Laplace approximation for definite integrals,** *J. Approx. Theory* **1,** *293–313 (1968).*

Key words: Asymptotic expansions; Bessel functions; definite integrals; error bounds; gamma function; Laplace approximation.

Explicit error bounds are obtained for the well-known asymptotic expansion of integrals of the form

$$\int_a^b e^{-\lambda p(x)} q(x)\, dx$$

in which λ is a large positive parameter, $p(x)$ and $q(x)$ are real differentiable functions, and $p'(x)$ has a simple zero in the finite or infinite range $[a, b]$. The bounds are expressed in terms of the supremum of a certain function taken over $[a, b]$, and are asymptotic to the absolute value of the first neglected term in the expansion as $\lambda \to \infty$. Several illustrative examples are given, including modified Bessel functions and the gamma function.

Other NBS Publications

J. Res. Nat. Bur. Stand. (U.S.), 73A (Phys. and Chem.) No. 5, 451–561 (Sept.–Oct. 1969), $1.50.

Conversion of existing calorimetrically determined thermodynamic properties to the basis of the International Practical Temperature Scale of 1968. T. B. Douglas.

Measured enthalpy and derived thermodynamic properties of alpha beryllium nitride, Be_3N_2, from 273 to 1200 K. T. B. Douglas and W. H. Payne.

Measured enthalpy and derived thermodynamic properties of solid and liquid lithium tetrafluoroberyllate, Li_2BeF_4, from 273 to 900 K. T. B. Douglas and W. H. Payne.

Nonanalytic vapor pressure equation with data for nitrogen and oxygen. R. D. Goodwin.

Some precise measurements of the vapor pressure of water in the range from 25 to 100 °C. H. F. Stimson.

The configurations $(3d+4s)^n 4p$ in neutral atoms of calcium, scandium, and titanium. C. Roth.

Absolute isotopic abundance ratio and atomic weight of terrestrial rubidium. E. J. Catanzaro, T. J. Murphy, E. L. Garner, and W. R. Shields.

J. Res. Nat. Bur. Stand. (U.S.), 73C (Eng. and Instr.) Nos. 1 & 2, 1–46 (Jan.–June 1969), $1.25.

An interferometer for measuring gradients in both refractive index and thickness of large or small optics. J. B. Saunders.

A Kerr electro-optical technique for observation and analysis of high-intensity electric fields. Esther C. Cassidy and Harold N. Cones.

Student-t deviate corresponding to a given normal deviate. Brian L. Joiner.

A heat loss compensated calorimeter and related theorems. Steve R. Domen.

Laboratory measurements of air cavity temperature in a passenger car tire. B. G. Simson and J. Mandel.

Bortner, M. H., A review of rate constants of selected reactions of interest in re-entry flow fields in the atmosphere, Nat. Bur. Stand. (U.S.), Tech. Note 484, 62 pages (May 1969), 60 cents.

Bullis, W. M., Editor, Methods of measurement for semiconductor materials, process control, and devices. Quarterly report January 1 to March 31, 1969, Nat. Bur. Stand. (U.S.), Tech. Note 488, 44 pages (July 1969), 50 cents.

Carter, R. S., Rush, J. J., Editors, Molecular dynamics and structure of solids. Proceedings of 2nd Material Research Symposium held at the National Bureau of Standards, Gaithersburg, Md., Oct. 16–19, 1967, Nat. Bur. Stand. (U.S.), Spec. Publ. 301, 571 pages (June 1969), $4.00.

Diamond, J. J., Editor, Bibliography on the high temperature chemistry and physics of materials, January, February, March 1969, Nat. Bur. Stand. (U.S.), Spec. Publ. 315–1, 81 pages (Apr. 1969), 75 cents.

Franklin, J. L., Dillard, J. G., Rosenstock, H. M., Herron, J. T., Draxl, K., Field, F., H., Ionization potentials, appearance potentials, and heats of formation of gaseous positive ions, Nat. Stand. Ref. Data. Ser., Nat. Bur. Stand. (U.S.), 26, 289 pages (June 1969), $4.00.

Koeser, R. L., Editor, Report of the 53d National Conference on Weights and Measures, 1969, Nat. Bur. Stand. (U.S.), Spec. Publ. 311, 182 pages (Mar. 1969), $1.25.

LaFleur, P. D. Editor, Activation Analysis Section: Summary of activities July 1967 to June 1968, Nat. Bur. Stand. (U.S.), Tech. Note 458, 109 pages (Mar. 1969), $1.00.

Lee, R. D., Construction and operation of a simple high-precision copper-point blackbody and furnace, Nat. Bur. Stand. (U.S.), Tech. Note 483, 16 pages (May 1969), 30 cents.

Little, J. L., Some evolving conventions and standards for character information coded in six, seven, and eight bits, Nat. Bur. Stand. (U.S.), Tech. Note 478, 30 pages (May 1969), 35 cents.

Mason, H. L., Editor, Making valuable measurements. Proceedings of the 1968 Standards Laboratory Conference, Nat. Bur. Stand. (U.S.), Spec. Publ. 313, 177 pages (May 1969), $1.50.

Moore, C. E., Bibliography on the analyses of optical atomic spectra. Section 3. ^{42}Mo-^{57} La, and ^{72}Hf-^{89} Ac, Nat. Bur. Stand. (U.S.), Spec. Publ. 306–3, 37 pages (May 1969), 50 cents.

Newman, N., Considerations in computing the useful frequency range of piezoelectric accelerometers, Nat. Bur. Stand. (U.S.), Tech. Note 487, 20 pages (July 1969), 30 cents.

Oberholtzer, B. L., Publications of the National Bureau of Standards 1966–1967, Nat. Bur. Stand. (U.S.), Spec. Publ. 305, 220 pages. (Apr. 1969), $2.00.

Rapial, A. S., Daney, D. E., Preparation and characterization of slush hydrogen and nitrogen gels, Nat. Bur. Stand. (U.S.), Tech. Note 378, 43 pages (May 1969), 50 cents.

Roberts, B. W., Superconductive materials and some of their properties, Nat. Bur. Stand. (U.S.), Tech. Note 482, 129 pages (May 1969), $1.25 Supersedes and extends Tech. Note 408.

Robinson, D. C., Misalinement detector for axial loading fatigue machines, Nat. Bur. Stand. (U.S.), Tech. Note 480, 15 pages (Apr. 1969), 25 cents.

Sindt, C. F., Ludtke, P. R., Daney, D. E., Slush hydrogen fluid characterization and instrumentation, Nat. Bur. Stand. (U.S.), Tech. Note 377, 69 pages (Feb. 1969), 65 cents.

Stabler, T. M., Editor, Specifications and tolerances for reference standards and field standard weights and measures. 1. Specifications and tolerances for field standard weights, Nat. Bur. Stand. (U.S.), Handb. 105–1, 10 pages (Apr. 1969), 25 cents.

Taggart, H. E., Workman, J. L., Calibration principles and procedures for field strength meters 30 (Hz to 1 GHz), Nat. Bur. Stand. (U.S.), Tech. Note 370, 157 pages (Mar. 1969), $1.25.

Wachtman, J. B., Jr., Editor, Mechanical and thermal properties of ceramics. Proceedings of a symposium Gaithersburg, Md., April 1–2, 1968, Nat. Bur. Stand. (U.S.), Spec. Publ. 303, 266 pages (May 1969), $4.25.

Wacker, P. F., Cord, M. S., Burkhard, D. G., Petersen, J. D., Kukol, R. F., Microwave spectral tables. Polyatomic molecules with internal rotation, Nat. Bur. Stand. (U.S.), Mono. 70, Volume III, 275 pages (June 1969), $4.25.

White, H. S., Jr., Editor, Implementation of the code for information interchange and related media standards. Hardware standards interchange codes and media, Nat. Bur. Stand. (U.S.), Fed. Info. Process. Stand. Publ. (FIPS Pub.) 7, 12 pages (Mar. 7, 1969), 25 cents. (Supplement to FIPS Pub. 1 through 3).

Acquista, N., Abramowitz, S., Lide, D. R., Structure of the alkali hydroxides. II. The infrared spectra of matrix isolated CsOH and CsOD, J. Chem. Phys. **49,** No. 2, 780–782 (July 15, 1968).

Ahearn, A. J., Paulsen, P. J., Workshop on mass spectrometric analysis of solids at the National Bureau of Standards in Gaithersburg, Md., Nov. 18 and 19, 1968, Anal. Chem. **41,** No. 2, 79A–81A (Feb. 1969).

Amano, T., Saito, S., Hirota, E., Morino, Y., Johnson, D. R., Powell, F. X., Microwave spectrum of the ClO radical, J. Mol. Spectry. **30,** No. 2, 275–289 (May 1969).

Anderson, H. J., Brenner, A., Preparation of rhenium hexafluoroacetylacetonate. J. Electrochem. Soc. **116,** No. 4, 513 (Apr. 1969).

Arenhovel, H., Danos, M., Baryon resonances in nuclei: Magnetic moment anomaly in ^{3}H and ^{3}He, Physics Letters **28B,** No. 5, 299–301 (Dec. 23, 1968).

Arp, V. D., Kasen, M. B., Reed, R. P., Magnetic energy storage and cryogenic aluminum amgnets, Tech. Rept. AFAPL–TR–68–87, 122 pages (Air Force Systems Command, Wright-Patterson Air Force Base, Ohio, Feb. 1969).

Ausloos, P., Rebbert, R. E., Lias, S. G., Gas-phase photolysis of cyclohexane in the photoionization region, J. Phys. Chem. **72,** No. 11, 3904–3914 (Oct. 1968).

Beatty, R. W., Discussion of effect of realizability conditions upon estimated limits of mismatch error in the calibration of fixed attenuators, IEEE Trans. Microwave Theory Tech. **MTT–16,** No. 11, 976 (Nov. 1968).

Bennett, H. S., Diffuse and propagating modes in the Heisenberg paramagnet, Phys. Rev. **174,** No. 2, 629–639 (Oct. 10, 1968).

Bennett, H. S., Phenomenology of neutron scattering in Heisenberg systems, Phys. Rev. **176,** No. 2, 650–654 (Dec. 10, 1968).

Berger, M. J., Seltzer, S. M., Chappell, S. E., Humphreys, J. C., Motz, J. W., Response of silicon detectors to monoenergetic electrons with energies between 0.15 and 5.0 MeV, Nucl. Instr. Methods **69,** No. 2, 181–193 (Apr. 1969).

Berman, H. A., West, E. D., Heat capacity of liquid nitromethane from 35° to 200° C, J. Chem. Eng. Data **14,** No. 1, 107–109 (Jan. 1969).

Birmingham, B. W., Flynn, T. M., National programs and the compressed gas industry, (Proc. 55th Annual Meeting of the Compressed Gas Association, New York City, N.Y., Jan. 1968), Chapter in The Future of the Compressed Gas Industry, pp. 39–49 (Compressed Gas Assoc. Inc., New York, N.Y., 1969).

Blanford, J. M., Bensing, P. L., Book, Testing Programs for the Apparel Industry. Evaluation of Material and Components, Part I, 31 pages (Oct. 1, 1968); Part II, 222 pages (Apparel Research Foundation, Inc., Washingtn, D.C., Oct. 1, 1968).

Boutillon, M., Henry, W. H., Lamperti, P. J., Comparison of exposure standards in the 10—50 kV X-ray region, Metrologia **5,** No. 1, 1–11 (Jan. 1969).

Bowen, R. L., Cleek, G. W., X-ray-opaque reinforcing fillers for composite materials, J. Dental Res. **48,** No. 1, 79–82 (Jan.–Feb. 1969).

Bowen, R. L., Mullineaux, A. L., Adhesive restorative materials, Dental Abstr. **14,** No. 80, 80–82 (Feb. 1969).

Branscomb, L. M., Joint Institute for Laboratory Astrophysics of the National Bureau of Standards and the University of Colorado, Boulder, Colorado, observatory report, Bull. Am. Astron. Soc. **1,** No. 1, 27–33 (Jan. 1969).

Broadhurst, M. G., The dielectric properties of leaves, sticks, and dirt at radio and microwave frequencies, Proc. 1968 Annual Report Conf. Electrical Insulation and Dielectric Phenomena, Oct. 21–23, 1968, Buck Hill Falls, Pa., pp. 146–152 (Natl. Acad. Sci.-Natl. Res. Council, Washington, D.C., 1969).

Brown, P. J., Human factors research in motor vehicle occupant restraint systems, IEEE Trans. Man-Machine Systems **MMS–9,** No. 3, 88–89 (Sept. 1969).

Bur, A. J., Roberts, D. E., Rodlike and random coil behavior of poly(*n*-butyl isocyanate) in dilute solution, Proc. 1968 Annual Report Conf. Electrical Insulation and Dielectric Pheonomena, Oct. 21–23, 1968, Buck Hill Falls, Pa., pp. 17–21 (Natl. Acad. Sci.-Natl. Res. Council, Washington, D.C., 1969).

Campbell, W. H., Aurora, Encyclopedia of Atmospheric Sciences and Astrogeology, R. W. Fairbridge, Ed., pp. 105–106 (Reinhold Publ. Co., New York, N.Y., 1967).

Casella, R. C., Time reversal and the K^0 meson decays. II, Phys. Rev. Letters **22,** No. 11, 554–556 (Mar. 17, 1969).

Cassidy, E. C., Cones, H. N., Wunsch, D. C., Booker, S. R., Calibration of a Kerr cell system for high-voltage pulse measurements, IEEE Trans. Instr. Meas. **IM–17,** No. 4, 313–320 (Dec. 1968).

Christ, B. W., Giles, P. M., On the detection of retained austenite in high-carbon steels by Fe^{57} Mössbauer spectroscopy, with appendix, Trans. Met. Soc. AIME **242,** 1915–1925 (Sept. 1968).

Clark, A. F., Fickett, F. R., A nonsuperconducting detection system for low level dc voltages, Rev. Sci. Instr. **40,** No. 3, 465–468 (Mar. 1969).

Clark, H. E., Young, R. D., Field emission through single strontium atoms adsorbed on a tungsten surface, Surfac Sci. **12,** 385–389 (1968).

Cole, A. R. H., Lafferty, W. J., Thibault, R. J., Rotation fine structure of the perpendicular band, ν_7, of ethane, J. Mol. Spectry. **29,** No. 3, 365–374 (Mar. 1969).

Coleman, J. A., Love, D. P., Trainor, J. H., Williams, D. J., Effects of damage by 0.8 MeV–5.0 MeV protons in silicon surface-barrier detectors, NASA Rept. No. x–611–68–221, 38 pages (National Aeronautics and Space Administration, Goddard Space Flight Center, Greenbelt, Md., June 1968).

Cooper, M. J., Point transformation of classical hard-core potential, J. Math. Phys. **9,** No. 4, 571–578 (Apr. 1968).

Cooper, M. J., Green, M. S., Generalized scaling and the critical eigenvector in ideal Bose condensation, Phys. Rev. **176,** No. 1, 302–309 (Dec. 5, 1968).

Coxon, B., P. M. R. evidence for the conformations of 1,2-0-benzylidene-α-D-glucofuranose derivatives, Carbohydrate Res. **8,** 125–125–1968).

Creitz, E. C., Construction details and measurements of response of a gas density balance, J. Chromatog. Sci. **7,** 137–144 (Mar. 1969).

Cutkosky, R. D., A varactor null detector for audio frequency capacitance bridges, IEEE Trans. Instr. Meas. **IM–17,** No. 4, 232–238 (Dec. 1968).

Czyz, W., Maximon, L. C., High energy, small angle elastic scattering of strongly interacting composite particles, Ann. Phys. **52,** No. 1, 59–121 (Mar. 1969).

Davis, D. D., Okabe, H., Determination of bond dissociation energies in hydrogen cyanide. Cyanogen and cyaongen halides by the photo-dissociation method, J. Chem. Phys. **49,** No. 12, 5526–5531 (Dec. 15, 1968).

Delaplane, R. G., Ibers, J. A., Ferraro, J. R., Rush, J. J., Diffraction and spectroscopic studies of the cobaltic acid system $HCoO_2$-$DCoO_2$, J. Chem. Phys. **50,** No. 5, 1920–1927 (Mar. 1, 1969).

Deslattes, R. D., Paretzkin, B., A small set of reference crystals elements and their compounds, Acta. Cryst. **A25,** Part 1, 89–93 (Jan. 1969).

Deslattes, R. D., Paretzkin, B., A small set of reference crystals for double-crystal topography, J. Appl. Cryst. 1, Part 3, 176–178 (Sept. 1968).

Dibeler, V. H., Liston, S. K., Mass-spectrometric study of photo-ionization, XI. Hydrogen sulfide and sulfur dioxide, J. Chem. Phys. **49,** No. 2, 482–485 (July 15, 1968).

Dibeler, V. H., Walker, J. A., Mass spectrmetric study of photo-ionization. XIII. Boron trichloride and diboron tetrachloride, Inorg. Chem. **8,** No. 1, 50-55 (Jan. 1969).

Drechsel, D., Maximon, L. C., Potential model calculation for coplanar and noncoplanar proton-proton bremsstrahlung, Ann. Phys. **49,** No. 3, 403–444 (Oct. 1968).

Driver, L. D., Arthur, M. G., A wideband RF voltmeter-comparator, IEEE Trans. Instr. Meas. **IM-17,** No. 2, 146–150 (June 1968).

Duerst, R. W., Baum, S. J., Kokoszka, G. F., Exchange coupling in two dimeric copper adenine complexes, Nature **222,** 665–666 (May 17, 1969).

Dunn, G. H., Colliding beams, (Proc. First Intern. Conf. Atomic Physics, June 3–7, 1968, New York, N.Y.), Chapter in Atomic Physics, B. Bederson, V. W. Cohen, F. M. J. Pichanich, Ed., pp. 417–433 (Plenum Press Inc., New York, N.Y., 1969).

Edelman, S., Roth, S., Grisham, L., Electrical generation of motion in elastomers, Shock and Vibration Bull. **39,** Part 2, 18 pages (Shock and Vibration Information Center, Naval Research Laboratory, Washington, D.C., Feb. 1969).

Edmonds, J., Matroid partition, Chapter in Mathematics of the Decision Sciences, American Mathematical Society Lectures in Applied Mathematics 11, 335–345 (American Mathematical Society, Providence, R.I., 1968).

Ehrlich, M., Influence of irradiation rate on the production of F centers in LiF (TLD grade), J. Appl. Phys. **40,** No. 2, 891–892 (Feb. 1969).

Ehrlich, M., Personal dosimetry: methods other than T.L.D., Proc. First Intern. Congress Radiation Protection, Rome, Italy, Sept. 5–10, 1966, Part 1, pp. 69–76 (Pergamon Press Inc., New York, N.Y., 1968).

Ehrlich, M., Lamperti, P. J., Uniformity of high-energy electron-beam calibrations, Phys. Med. Biol. **14,** No. 2, 305–314 (Sept. 1968).

Eick, J. D., Hegdahl, T., Segregation in dental gold casting alloys, J. Dental Res. **47,** No. 6, Part 2, 1118–1127 (Nov.–Dec. 1968).

Eicke, W. G., The zener diode, a working D-C voltage standard, 1968 IEEE Intern. Convention Digest, page 71 (Mar. 1968).

Enemark, E. A., Gallagher, A., A pockels cell light modulator for wide angle radiation, Rev. Sci. Instr. **40,** No. 1, 40–41 (Jan. 1969).

Engen, G. F., A method of determining the mismatch correction in microwave power measurements, IEEE Trans. Instr. Meas. **IM-17,** No. 4, 392–395 (Dec. 1968).

Ensign, T. C., Chang, T. T., Low temperature optical-EPR sample probe using tunable coaxial coupling, Rev. Sci. Instr. **40,** No. 2, 268–270 (Feb. 1969).

Ernst, M. H., Transport coefficients from time correlation functions, Lectures Theoret. Phys. IX C-Kinetic Theory, 417–441 (Gordon and Breach, Science Publ. Inc., New York, N.Y., 1967).

Farrar, T. C., Johannesen, R. B., Coyle, T. D., Magnetic nonequivalence in the high-resolution NMR spectra of diborane, J. Chem. Phys. **49,** No. 1, 281–285 (July 1, 1968).

Fatiadi, A. J., Evidence for a chair conformation of certain osotriazoles of inositols, Chem. Ind. **19,** 617–619 (May 10, 1969).

Freeman, D. H., Paulson, R. A., Chemical microstandards from ion exchange resin, Nature **218,** 563 (1968).

Frisch, R. C., Forman, R. A., Nuclear magnetic resonance of Ti in Ti-H systems, J. Chem. Phys. **48,** No. 11, 5187–5190 (June 1, 1968).

Gadsden, M., Airglow, Encyclopedia of Atmospheric Sciences and Astrogeology, R. W. Fairbridge, Ed., pp 8–10 (Reinhold Publ. Co., New York, N.Y., 1967).

Gadsden, M., Airglow motions, International Dictionary of Geophysics **1,** 50–56 (1967).

Gallagher, A., Excitation transfer in nearly adiabatic thermal collisions, Phys. Rev. **179,** No. 1, 105–110 (Mar. 5, 1969).

Gardner, T. V., Jr., Dickson, G., Kumpula, J. W., Application of diffraction gratings to measurement of strain of dental materials, J. Dental Res. **47,** No. 6, Part 2, 1104–1110 (Nov.–Dec. 1968).

Garfinkel, S. B., Mann, W. B., A method for obtaining large numbers of measured time intervals in radioactive decay, Intern. J. Appl. Radiation Isotopes **19,** No. 9, 707–709 (Sept. 1968).

Gatterer, L. E., Bottone, P. W., Morgan, A. H., Worldwide clock synchronization using a synchronous satellite, IEEE Trans. Instr. Meas. **IM-17,** No. 4, 372–378 (Dec. 1968).

Gautier, T. N., Zacharisen, D. H., Use of space and time correlations in short-term ionospheric predictions, First Ann. IEEE Commun. Convention, pp. 671–676 (June 1965).

Geltman, S., Topics in atomic collision theory, 30, 247 pages (Academic Press Inc., New York, N.Y., 1969).

Ginnings, D. C., Introduction in Experimental Thermodynamics, J. P. McCullough and D. W. Scott, Ed., Vol. I, Calorimetry of Non-reacting Systems, Chapt. 1, 1–13 (Butterworth & Co., London, England, 1968).

Ginnings, D. C., Stimson, H. F., Calorimetry of saturated fluids including determination of enthalpies of vaporization, Chapter in Expirmental Thermodynamics, J. P., McCullough and D. W. Scott, Ed., Vol. **I,** Calorimetry of Non-reacting Systems, Chapt. 11, 395–420 (Butterworth & Co., London, England, 1968).

Ginnings, D. C., West, E. D., Principles of calorimetric design, Chapter in Experimental Thermodynamics, J. P., McCullough and D. W. Scott, Ed., Vol **I,** Calorimetry of Non-reacting Systems, Chapt. 4, 85–131 (Butterworth & Co., London, England, 1968).

Glasgow, A. R., Jr., Ross, G. S., Cryoscopy, Chapter 88 in Treatise on Analytical Chemistry, Part I, Theory and Practice, I. M. Kolthoff, P. J. Elving, E. B. Sandell, Ed., **8,** 4991–5083 (Interscience Publ., New York, N.Y., 1968).

Golub, S., Steiner, B., Photodetachment of $[OH(H_2O)]^-$, J. Chem. Phys. **49,** No. 11, 5191–5192 (Dec. 1, 1968).

Grabner, L., Photoluminescence in $SrTiO_3$, Phys. Rev. **177,** No. 3, 1315–1323 (Jan. 15, 1969).

Hanley, H. J. M., Childs, G. E., Interim values for the viscosity and thermal conductivity coefficients of fluid He^4 between 2 and 50 K, Cryogenics **9,** No. 2, 106–111 (Apr. 1969).

Hanley, H. J. M., McCarty, R. D., Sengers, J., V., Density dependence of experimental transport coefficients of gases, J. Chem. Phys. **50,** No. 2, 857–870 (Jan. 15, 1969).

Hardy, S. C., Coriell, S. R., Morphological stability and the ice-water interfacial free energy, J. Crystal Growth **3,** No. 4, 569–573 (1968).

Harris, W. P., Roberts, D. E., New standard reference dielectric specimens at NBS, Proc. 1968 Annual Report Conf. Electrical Insulation and Dielectric Phenomena, Oct. 21–23, 1968, Buck Hill Falls, Pa., pp. 80–84 (Natl. Acad. Sci.-Natl. Res. Council, Washington, DC. 1969).

Hartman, W. J., Feasibility of communication with a satellite by means of tropospheric scatter, IEEE Trans. Commun. Technol. **COM-14,** No. 3, 251 (June 1966).

Heinrich, K. F. J., Common sources of error in electron probe micro-analysis, Advan. X-ray Anal. **11,** 40–55 (1968).

Henderson, M. M., Introductory remarks and summary of discussion for symposium on critical reviews, J. Chem. Doc. **8,** 231 and 245 (Nov. 1968).

Herron, J. T., Penzhorn, R. D., Mass spectrometric study of the reactions of atomic oxygen with ethylene and formaldehyde, J. Phys. Chem. **73,** No. 1, 191–196 (Jan. 1969).

Hoover, T. B., The *N*-methylpropionamide-water system. Densities and dielectric constants at 20–40°, J. Phys. Chem. **73,** No. 1, 57–61 (Jan. 1969).

Hiza, M. J., Duncan, A. G., A simple gas recirculation pump for low flow and high pressure applications, Rev. Sci. Instr. **40,** No. 3, 513–514 (Mar. 1969).

Hubbell, J. H., Berger, M. J., Photon attenuation. Attenuation coefficients, energy absorption coefficients, and related quantities, Chapter 4 in Engineering Compendium on Radiation Shielding, R. G. Jaeger, E. P. Blizard, A. B. Chilton, M. Grotenhuis, A. Honig, T. A. Jaeger, H. Eisenlohr, Eds. 1, 167–202 (International Atomic Energy Agency, Vienna, Austria, 1968).

Hudson, R. P., The delta campaign. An account of the controversy surrounding the temperature scale for CMN, Cryogenics **9,** No. 2, 76–79 (Apr. 1969).

Hummer, D. G., Rybicki, G. B., Line formation in differentially moving media with temperature gradients, Proc. Natl. Center for Atmospheric Research Conf. Resonance Lines in Astrophysics,

Sept. 9–12, 1968, Boulder, Colo., pp. 315–330 (National Center for Atmospheric Research, Boulder, Colo., Dec. 1968).

Ito, J., Johnson, H., Synthesis and study of yttrialite, Am. Mineralogist **53, 1940**–1952 (1968).

Ives, J. M., Hughes, E. E., Taylor, J. K., Absolute determination of low concentrations of oxygen in inert gases by means of galvanic cells, Anal. Chem. **40,** No. 12, 1853–1856 (Oct. 1968).

Judd, D. B., 1964 CIE Supplementary observer applied to the colorimetry of rutile and anatase forms of titanium dioxide, J. Opt. Soc. Am. **58,** No. 12, 1638–1649 (Dec. 1968).

Kamper, R. A., Cryoelectronics, Cryogenics **9,** No. 1, 20–25 (Feb. 1969).

Kaufman, V., Radziemski, L. J., Redetermined level values and interaction parameters of Si I and Cl II ground configurations, J. Opt. Soc. Am. **59,** No. 2, 227–228 (Feb. 1969).

Keller, R. A., Excited triplet→ singlet intersystem crossing, Chem. Phys. Letters **3,** No. 1, 27–29 (Jan. 1969).

King, R. J., Maley, S. W., Wait, J. R., Experimental and theoretical studies of propagation of ground waves across mixed paths (Proc. URSI Symp. Electromagnetic Wave Theory, Delft, Holland, Sept. 1965), Chapter in Electromagnetic Wave Theory, J. Brown, Ed., pp. 217–224 (Pergamon Press, Inc., New York, N.Y., 1965).

Klose, J. Z., Experimental lifetimes of the 5*p* levels in argon I, J. Opt. Soc. Am. **58,** No. 11, 1509–1512 (Nov. 1968).

Koonce, C. S., Cohen, M. L., Theory of superconducting semiconductors and semimetals, Phys. Rev. **177,** No. 2, 707–719 (Jan. 10, 1969).

Kraft, R., An unusual mixed initial boundary problem in two independent variables, J. Math. Anal. Appl. **25,** No. 2, 233–249 (Feb. 1969).

Kropschot, R. H., Report on the fourteenth cryogenic engineering conference, Cryogenics **8,** No. 6, 346–348 (Dec. 1968).

Lafferty, W. J., Maki, A. G., Pringle, W. C., Jr., Comments on "Two rotational lines of allene C_3H_4", J. Chem. Phys. **50,** No. 1, 564–565 (Jan. 1, 1969).

Lance, H. W., Metrology and National goals (Proc. 22nd ISA Conf. and Exhibit, Chicago, Ill., Sept. 11–14, 1967) ISA Preprint No. M4–1–MESTIND–67, 1–14 (1967); excerpt in Meas. Data **2,** No. 2, 68–71 (Mar.–Apr. 1968).

Lang, S. B., Shaw, S. A., Rice, L. H., Timmerhaus, K. D., Pyroelectric thermometer for use at low temperatures, Rev. Sci. Instr. **40,** No. 2, 274–284 (Feb. 1969).

Lashof, T. W., Standardization and application of accuracy and precision statements: TAPPI Standards Committee-ASTM Committee D-6 policy, Tappi **4,** 732–739 (Apr. 1969).

Latanision, R. M., Staehle, R. W., Plastic deformation of electrochemically polarized nickel single crystals, Acta Met. **17,** 307–319 (Mar. 1969).

Lee, T. G., A system for continuously monitoring hydrogen chloride concentrations in gaseous mixtures using a chloride ion-selective electrode, Anal. Chem. **41,** 391–392 (Feb. 1969).

Levin, E. M., X-ray determination of the thermal expansion of silver nitrate, J. Am. Ceramic Soc. **52,** No. 1, 53–54 (Jan. 1969).

Linzer, M., Line parameters of Lorentzian curves containing arbitrary mixtures of absorption and dispersion, J. Appl. Phys. **40,** No. 1, 372–376 (Jan. 1969).

Lloyd, E. C., Standards for improved measurement of pressure and vacuum (Proc. 4th Intern. Measurement Conf., Warsaw, Poland, July 2–7, 1967), Acta Imeko, pp. 451–468 (Hungarian Academy of Sciences, Budapest, Hungary, 1967).

Lloyd, E. C., Beckett, C. W., Boyd, F. R., Jr., Measurements in high-pressure environment, Science **164,** 860–861 (May 16, 1969).

Lutz, G. J., Calculation of sensitivities in photon activation analysis, Anal. Chem. **41,** No. 3, 424–427 (Mar. 1969).

McIntyre, D., Sengers, J. V., Study of fluids by light scattering, Chapter 11 in Physics of Simple Liquids, H. N. V. Temperley, J. S. Rowlinson, G. S. Rushbrooke, Eds., pp. 449–505 (North Holland Publ. Co., Amsterdam, The Netherlands, 1968).

McKinley, J. D., Jr., Mass spectrometric investigation of the surface reaction of tungsten with chlorine-oxygen mixtures, (Proc. 6th Intern. Symp. Reactivity of Solids, Schenectady, N.Y., Aug. 25–30, 1968), Chapter in Reactivity of Solids, J. W. Mitchell, R. C. DeVries, R. W. Roberts, and P. Cannon, Eds., pp 345–351 (John Wiley & Sons, Inc., New York, N.Y., 1969).

Madden, R. P., Ederer, D. L., Codling, K., Resonances in the photoionization continuum of Ar I (20–150 eV), Phys. Rev. **177,** No. 1, 136–151 (Jan. 5, 1969).

Maki, A. G., Comment on the fermi resonance constant in ICN, J. Chem. Phys. **50,** No. 5, 2273 (Mar. 1, 1969).

Marinenko, G., Taylor, J. K., Electrochemical equivalents of benzoic and oxalic acid, Anal. Chem. **40,** No. 11, 1645–1651 (Sept. 1968).

Matsumura, C., Lide, D. R., Jr., Structure of the alkali hydroxides. III. Microwave spectra of RbOH and RbOD, J. Chem. Phys. **50,** No. 1, 71–75 (Jan. 1, 1969).

Mighell, A. D., Reimann, C. W., Mauer, F. A., The crystal and molecular structure of diaquobis-(2,2'-biimidazole)nickel(II) dinitrate, $Ni(C_6H_6N_4)_2(H_2O)_2(NO_3)_2$, Acta Cryst. **B25,** Part 1, 61–66 (Jan. 1969).

Mighell, A. D., Reimann, C. W., Santoro, A., The crystal and molecular structure of dibromotetrapyrazolenickel (II), $Ni(C_3H_4N_2)_4Br_2$, Acta Cryst. **B25,** Part 3, 595–599 (Mar. 1969).

Mills, R. M., Flame inhibition with electron attachment as the first step, Combustion and Flame **12,** No. 2, 513–520 (Dec. 1968).

Moore, C. E., Annual report on spectroscopy, Bull. Am. Astron. Soc. **1,** No. 1, 85–86 (Jan. 1969).

Mopsik, F. I., Dielectric properties of slightly polar organic liquids as a function of pressure, volume, and temperature, J. Chem. Phys. **50,** No. 6, 2559–2569 (Mar. 15, 1969).

Norton, R. B., Aeronomy, Encyclopedia of Atmospheric Sciences and Astrogeology, R. W. Fairbridge, Ed., pp. 3–7 (Reinhold Publ. Co., New York, N.Y., 1967).

O'Reilly, D. E., Tsang, T., Order-disorder transition in ferroelectric ammonium sulfate, J. Chem. Phys. Letter to Editor **50,** No. 5, 2274–2275 (Mar. 1, 1969).

Paabo, M., Bates, R. G., Standards for a practical scale of pD in heavy water, Anal. Chem. **41,** No. 2, 283–285 (Feb. 1969).

Plummer, E. W., Gadzuk, J. W., Young, R. D., Resonance tunneling of field emitted electrons through adsorbates on metal surfaces, Solid State Commun. **7,** 487–491 (1969).

Pool, K. H., Bates, R. G., Thermodynamics of hydrochloric acid in 95 volume percent (92.41 mass percent) ethanol from e.m.f. measurements at 5 to 50 °C, J. Chem. Thermodynamics **1,** 21–29 (1969).

Powell, R. L., Clark, A. F., Fickett, F. R., Longitudinal magnetoresistance of pure copper (Proc. Conf. Electron Mean Free Paths in Metals, Zurich, Switzerland, Sept. 3–5, 1968), Phys. kondens. Materie **9,** 104–112 (Lange and Springer, Berlin, West Germany, 1969).

Quindry, T. L., Transmissibility measurements for the determination of structural damping (Proc. Vibrations Conf., Philadelphia, Pa., Mar. 30–Apr. 2, 1969), Preprint No. 69–VIBR–36, 7 pages (American Society of Mechanical Engineers, New York, N.Y., 1969).

Rebbert, R. E., Ausloos, P., The photolysis of methane at 584 Å, J. Am. Chem. Soc. **90,** 7370–7371 (1968).

Risley, E. W., Discontinuity capacitance of a coaxial line terminated in a circular waveguide, IEEE Trans. Microwave Theory and Tech. **MTT–17,** No. 2, 86–92 (Feb. 1969).

Roach, F. E., Diffuse galactic light, Chapter in Modern Astrophysics: A Memorial to Otto Struve, M. Hack, Ed., pp. 49–66 (Gordon and Breach, New York, N.Y., 1967).

Roach, F. E., The brightness of the zodical light at the ecliptic pole (Proc. Symp. Meteor Orbits and Dust, Cambridge, Mass., Aug. 9–19, 1965), Smithsonian Contributions to Astrophysics, F. L. Whipple, Ed., 11, 225–226 (Smithsonian Institution Astrophysics Observatory, Cambridge, Mass., 1967).

Ryan, J. V., Flammable fabrics act and its implementation, Mod. Textiles Mag. **L,** No. 3, 67–70 (Mar. 1969).

Sanderson, B. T., Romanoff, M., Performance of commercially pure titanium in corrosive soils, Mater. Production 8, No. 4, 29–32 (Apr. 1969).

Sarachman, T. N., Microwave spectrum of *cis*-2-butene, J. Chem. Phys. **49,** No. 7, 3146–3149 (Oct. 1, 1968).

Scala, A. A., Ausloos, P., Gas-phase photolysis and radiolysis of cyclopropane, J. Chem. Phys. **49,** No. 5, 2282–2290 (Sept. 1, 1968).

Schafer, G. E., Precision microwave measurements—today and tomorrow, Microwave J. **12,** No. 6, 25 (June 1969).

Shah, J. A., Huber, M. G., Danos, M., Separability of the nuclear residual two-body interaction, Physics Letters **28B,** No. 6, 381–383 (Jan. 6, 1969).

Shirk, J. S., Bass, A. M., Matrix-isolation spectra of discharge "sputtered" metals, J. Chem. Phys. **49,** No. 11, 5156–5160 (Dec. 1, 1968).

Shirley, J. H., Dynamics of a simple maser model, Am. J. Phys. **36,** Part 1, No. 11, 949–963 (Nov. 1968).
Sieck, L. W., Xenon-photosensitized decomposition of methane at 1470 A, J. Chem. Phys. **50,** No. 4, 1748–1753 (Feb. 15, 1969).
Smith, R. V., Current developments in cryomedicine, Cryogenics **9,** No. 2, 84–89 (Apr. 1969).
Smith, R. V., Review of heat transfer to helium I, Cryogenics **9,** No. 1, 11–19 (Feb. 1969).
Smith, S. J., Photodetachment, Chapter in Methods in Experimental Physics, Vol. **7,** Atomic and Electron Physics, B. Bederson and W. L. Fite, Eds., pp. 179–208 (Academic Press Inc., New York, N. Y., 1968).
Steiner, B., Photodetachment of electrons from SH^-, J. Chem. Phys. **49,** No. 11, 5097–5104 (Dec. 1, 1968).
Stephens, R. E., Sutton, L. E., Diffraction image of a point in the focal plane and several out-of-focus planes, J. Opt. Soc. Am. **58,** No. 7, 1001–1002 (July 1968).
Stiehler, R. D., Parks, E. J., Linnig, F. J., Stiffening of elastometers by organic fillers, Appl. Poly. Symp. No. 7, 143–153 (1968).
Swartzendruber, L. J., Variable temperature Mössbauer effect spectrometer with provision for longitudinal external fields, Nucl. Instr. Methods **69,** 101–105 (1969).
Swartzendruber, L. J., Ulmer, F. H., Coleman, J. A., Direct reading instrument for silicon and germanium four-probe resistivity measurements, Rev. Sci. Instr. **39,** No. 12, 1858–1863 (Dec. 1968).
Tipson, R. S., West, B. F., Brady, R. F., Acid-catalyzed hydrolysis of isopropylidene acetals of some 2-pentuloses and 2-hexuloses, Carbohydrate Res. **10,** 181–183 (1969).
Trembath, C. L., Wait, D. F., Engen, G. F., Foote, W. J., A low-temperature microwave noise standard, IEEE Trans, Microwave Theory Tech. **MTT**-16, No. 9, 709–714 (Sept. 1968).
Tsang, T., Self-consistent correlation function approximation in ising ferromagnets, Physica **42,** 1–11 (1969).
Tsang, T., Farrar, T. C., Nuclear magnetic relaxation studies of internal rotations and phase transitions in borohydrides of lithium, sodium, and potassium, J. Chem. Phys. **50,** No. 8, 3498–3502 (Apr. 15, 1969).
Tsang, W., Thermal decomposition of 3,4-dimethylpentene-1,2,3,3-trimethylpentane, 3,3-dimethylpentane, and isobutylbenzene in a single pulse shock tube, Intern. J. Chem. Kinetics **1,** 245–278 (1969).
Waterstrat, R. M., Evaluation of a gallium-palladium-tin alloy for restorative dentistry, J. Am. Dental Assoc. **78,** No. 3, 536–541 (Mar. 1969).
Weir, C. E., Piermarini, G. J., Block, S., Crystallography of some high-pressure forms of C_6H_6, CS_2, Br_2, CCl_4, and KNO_3, J. Chem. Phys. **50,** No. 5, 2089–2093 (Mar. 1, 1969).
Weisman, I. D., Swartzendruber, L. J., Bennett, L. H., Resonance studies in ferromagnetic Fe_2B and Fe_2Zr, Phys. Rev. **177,** No. 2, 465–471 (Jan. 10, 1969).
Weiss, A. W., Series perturbations and atomic oscillator strengths: The ^2D series of A II, Phys. Rev. **178,** No. 1, 82–89 (Feb. 5, 1969).
Weiss, B-Z., Meyerson, M. R., The effect of mean stress on fatigue crack propagation in plates under extension and bending, J. Basic Eng., **90,** Series D, No. 3, 414–417 (Sept. 1968).
Weissberg, S. G., Brown, J. E., Osmometry of polymers, Encyclopedia of Polymer Science and Technology **9,** 659–668 (interscience Publs., New York, N.Y., 1969).
White, H. S., Jr., Data management and standardization, Mod. Data **2,** No. 5, 52–56 (May 1969).
Wiederhorn, S. M., Fracture surface energy of glass, J. Am. Ceramic Soc. **52,** No. 2, 99–105 (1969).
Wiese, W. L., Systematic trends of atomic oscillator strengths in isoelectronic sequences, Appl. Opt. **7,** No. 12, 2361–2366 (Dec. 1968).
Wims, A. M., McIntyre, D., Hynne, F., Coexistence curve for 3-methylpentane-nitroethane near the critical point, J. Chem. Phys. **50,** No. 2, 616–620 (Jan. 15, 1969).
Wolfgang, R., Zare, R. N., Branscomb, L. M., Chemical accelerators, Science **162,** No. 3855, 818–822 (Nov. 15, 1968).

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